Module 3: Digital Modulation Techniques

When digital data is transmitted over bandpass channel, it is necessary to modulate the incoming data onto a carrier wave with fixed frequency imposed by the channel. This modulation process involves switching/keying the amplitude, frequency & phase of the carrier in accordance with the incoming data.

The basic modulation technique for the transmission of digital data are:

* Amplitude shift keying [ASK]
* Frequency shift keying [FSK]
* Phase shift keying [PSK]

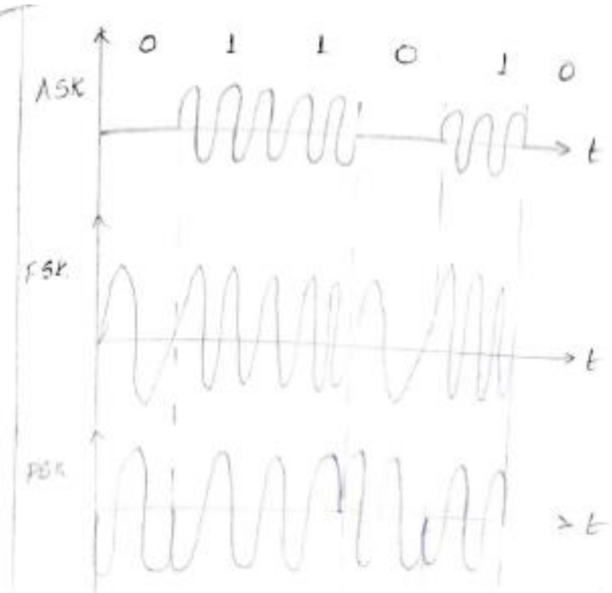
These are the special cases of AM, FM & PM.

* Digital modulation formats :-

→ The process of varying the characteristics of a carrier in accordance with a modulating wave is called modulation.

→ The modulating wave used in digital communication consist of binary data & the carrier is sinusoidal wave.

→ the feature used by modulator to discriminate one signal from another is a step change in amplitude, frequency (or) phase, thus the



→ FSK & PSK are widely used than ASK.

→ Sometimes there can be hybrid modulation which is nothing but change in both amplitude & phase, such combination is called "amplitude phase keying." [APK]

→ At receiver end, the demodulation can be coherent (or) noncoherent detection

→ In coherent detection, the receiver should also have the carrier waves phase reference that is provided at transmitter.

→ Coherent detection is performed as follows,

i. Cross correlation of received signal with carrier.

ii. Decision making based on threshold value.

→ In noncoherent detection, the receiver does not require the information wrt carrier wave phase, thereby receiver complexity is reduced but error is introduced

→ The choice of different modulation scheme are based on the following requirements,

* Minimum probability of symbol error.
* Minimum transmission power
* Minimum channel B.W.
* Maximum resistance to interfering signals.
* Minimum circuit complexity.

* **Coherent binary modulation techniques:-**

→ The 3 basic forms of binary modulation techniques are:
  * Amplitude shift Keying
  * Frequency shift Keying
  * phase shift Keying.

→ The noise analysis of coherent detection of ASK, PSK & FSK is briefly explained by assuming additive white gaussian noise model [AWGN]

→ Signal constellation is a set of possible message point

→ Constellation diagram represents a signal as a 2D scattered diagram on a complex plane at the sampling instants. It helps us to recognize the type of interference in a signal

* **Phase shift keying techniques using coherent detection:-**

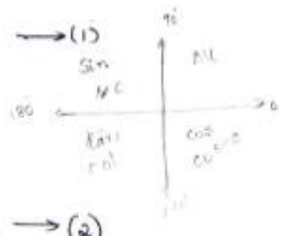
1. **Binary phase shift keying:-**

In a binary PSK, the pair of signals $s_1(t)$ & $s_2(t)$ used to represent binary symbols '1' & '0' respectively is used as,

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos 2\pi f_c t \qquad 0 \le t \le T_b \longrightarrow (1)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \pi)$$

$$s_2(t) = -\sqrt{\frac{2E_b}{T_b}}\cos 2\pi f_c t \qquad 0 \le t \le T_b \longrightarrow (2)$$



Where, $T_b \rightarrow$ symbol interval

$E_b \rightarrow$ Transmitted signal energy per bit.

- carrier frequency is choosen equal to $\frac{n_c}{T_b}$; where, $n_c \rightarrow$ any integer

- from eqn (1) & (2), $s_1(t)$ & $s_2(t)$ are out of phase by 180°, is referred as "antipodal signal"

→ The only basis function $\phi_1(t)$ of unit energy is,

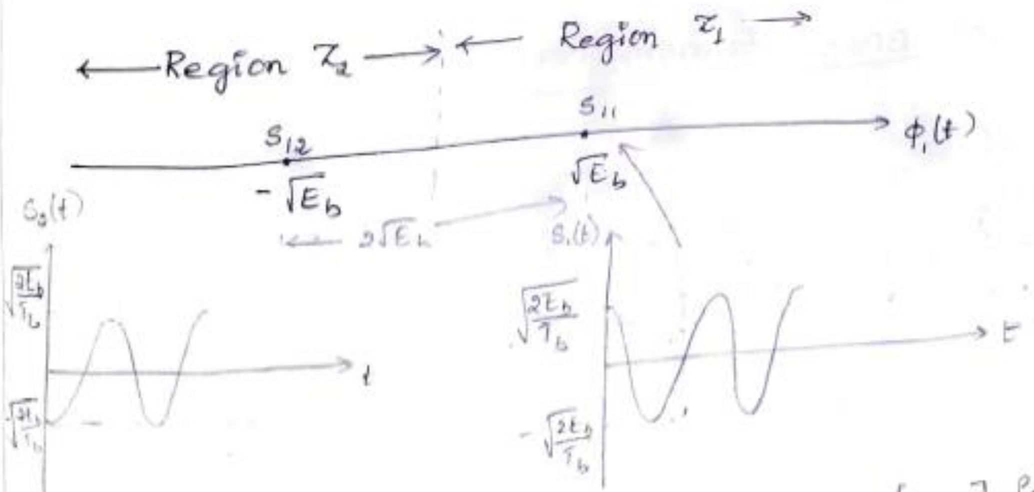
$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_c t \qquad 0 < t < T_b \longrightarrow (3)$$

- Expressing $s_1(t)$ & $s_2(t)$ interms of $\phi_1(t)$,

$$s_1(t) = \sqrt{E_b}\, \phi_1(t) \qquad 0 \leq t \leq T_b \longrightarrow (4)$$

$$s_2(t) = -\sqrt{E_b}\, \phi_1(t) \qquad 0 \leq t \leq T_b \longrightarrow (5)$$

∴ coherent BPSK is characterized by one dimension with two message points. The message point corresponding to $s_1(t)$ is at $+\sqrt{E_b}$ & $s_2(t)$ is at '$-\sqrt{E_b}$'. The signal Constellation diagram is shown in fig. below



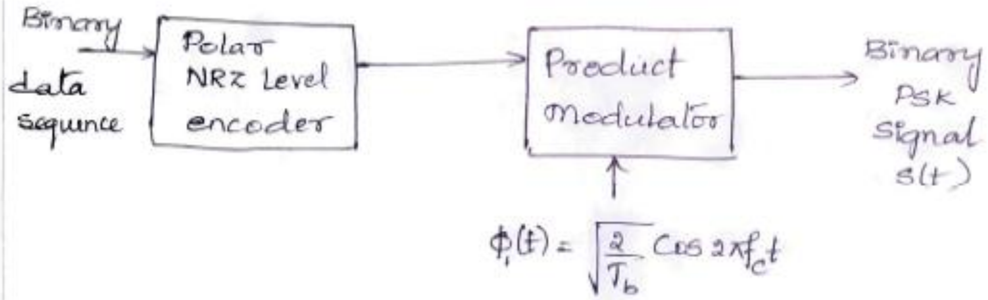
Signal Space diagram & waveforms $[n_c = 2]$ for BPSK

→ Assuming symbol '1' & '0' occurs with equal probabili if set of points reside closer to 'S_{11}' then it corresponds to symbol '1' transmission & if set of points are closer to 'S_{21}' then it corresponds to symbol '0' transmission.

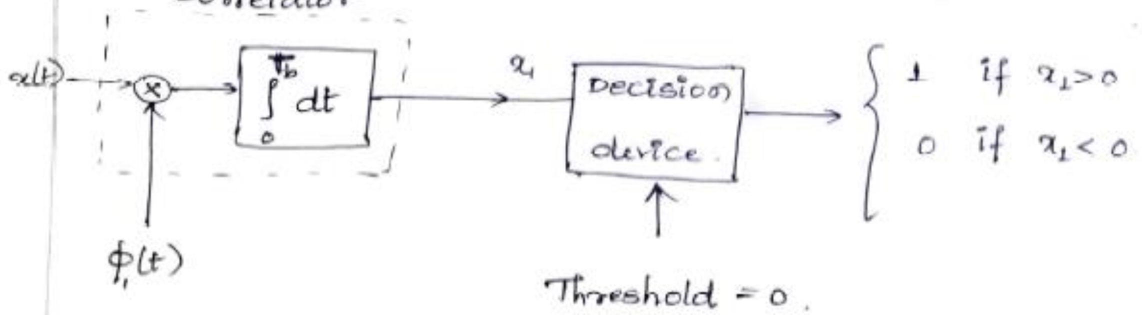
→ The distance b/w the two message point is $2\sqrt{E_b}$

→ Error occurs when signal 'S_2' is transmitted, but due to noise received signal falls in region Z_1 & when 'S_1' is transmitted but if the received signal falls in region 'Z_2'

* Functional schematic of PSK generation :-



(a) BPSK transmitter.



(b) BPSK receiver

→ The i/p binary sequence is polar NRZ format. Symbol '1' is represented by rectangular pulse of constant amplitude '$+\sqrt{E_b}$' & symbol '0' by '$-\sqrt{E_b}$'

→ second i/p to product modulator is $\phi_1(t)$ & o/p of product modulator is binary PSK signal $s(t)$.

→ Since the information resides in the phase of carrier, phase reference must be present at receiver end. Hence this detection process is called coherent detection.

→ At the Rxr end, $x(t)$ is the received signal which includes AWGN. $\phi_1(t)$ is synchronized w.r.t. phase & frequency of carrier at Txr.

→ The two basic components of PSK Rxr are,

1. Correlator :- which correlates $x(t)$ with $\phi_1(t)$ on a bit-by-bit basis.

2. Decision device :- It compares the correlator o/p with the zero threshold.
   * If $x_1 > 0$, then decision is in favour of symbol '1'
   * If $x_1 < 0$, then decision is in favour of symbol '0'
   * If $x_1 = 0$, then decision is arbitrary.

Probability of error calculation :-

Let $x(t)$ be received signal,

$$x(t) = s(t) + w(t) \qquad 0 \le t \le T_b$$

Where, $w(t) \rightarrow$ AWGN.

Assuming symbol '0' (or) s_2 is transmitted, then the T_b

$$\therefore x_1(t) = \int_0^{T_b} [s_2(t) + w(t)]\,\phi_1(t)\,dt = \int_0^{T_b} s_2(t)\,\phi_1(t)\,dt + \int_0^{T_b} w(t)\phi_1(t)\,dt$$

$$x_1 = s_{21} + w_1 \longrightarrow (1)$$

But $s_{21} = -\sqrt{E_b}$

$$\therefore x_1 = -\sqrt{E_b} + w_1 \longrightarrow (2)$$

Where, $w_1 \rightarrow$ sample value of random variable W_1 with mean = 0 & variance, $\sigma^2 = \frac{N_0}{2}$

$x_1 \Rightarrow$ sample value of gaussian random variable 'X_1'.

$$\therefore E[X_1] = E\left[-\sqrt{E_b} + W_1\right] = -\sqrt{E_b} + E[W_1]$$

$$\mu = -\sqrt{E_b} + 0$$

$$\hookrightarrow = -\sqrt{E_b} \longrightarrow (3)$$

Variance of X_1 is, $Var[X_1] = Var[-\sqrt{E_b}] + Var[W_1]$

W.K.T. Variance of constant is zero

$$\therefore Var[X_1] = 0 + \frac{N_0}{2} = \frac{N_0}{2} \rightarrow (4)$$

The conditional probability density function of random variable 'X_1' given that symbol '0' is

$$\therefore f_{x_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} e^{\left[\frac{-(x_1+\sqrt{E_b})^2}{N_0}\right]} \longrightarrow (5)$$

$\therefore$ Probability of error of '0' is $P_e(0)$ denotes the decision in favour of symbol '1' when '0' is transmitted.

$$P_e(0) = P\left[x_1 > 0 /_{\text{symbol '0' is transmitted.}}\right]$$

Region $Z_1 : 0 \leq x_1 \leq +\infty$

$$\therefore P_e(0) = \int_0^\infty f_{x_1}(x_1|0)\, dx_1$$

$$\hookrightarrow = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty e^{-\left[\frac{(x_1+\sqrt{E_b})}{\sqrt{N_0}}\right]^2} dx_1 \longrightarrow (6)$$

Let $\frac{x_1+\sqrt{E_b}}{\sqrt{N_0}} = z$.

$\therefore \frac{dx_1}{\sqrt{N_0}} = dz$

$dx_1 = \sqrt{N_0} \cdot dz$

When $x_1 = 0$, $z = \frac{\sqrt{E_b}}{\sqrt{N_0}}$

When $x_1 = \infty$, $z = \infty$

$$\therefore P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-z^2} \cdot \sqrt{N_0}\, dz = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-z^2} dz.$$

$\therefore$ Integral eqn. can be expressed interms of Complementary error function {erfc} then.

$$\boxed{P_e(0) = \frac{1}{2} \operatorname{erfc}\sqrt{\frac{E_b}{N_0}}} \longrightarrow (7)$$

Symbol 1:-

$$x_1 = \int_0^{T_b} [s(t) + w(t)] \phi_1(t)\, dt = \int_0^{T_b} s_1(t)\phi_1(t)\, dt + \int_0^{T_b} w(t)\phi_1(t)\, dt$$

$$\hookrightarrow = s_{11} + w_1$$

$$E[x_1] = E[s_{11}] + E[w_1] = \sqrt{E_b} + 0 = \sqrt{E_b}$$

$$Var[x_1] = Var[s_{11}] + Var[w_1] = 0 + \frac{N_0}{2} = \frac{N_0}{2}$$

$\therefore$ Conditional PDF is

$$f_{x_1}(x_1|1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}$$

$$\hookrightarrow = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x_1-\sqrt{E_b})^2}{N_0}}$$

$\therefore$ Probability of error, $P_e(1) = \int_{-\infty}^{0} f_{x_1}(x_1|1)\, dx_1$

$$P_e(1) = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x_1-\sqrt{E_b})^2}{N_0}}\, dx_1$$

$$\hookrightarrow = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-\left[\frac{x_1-\sqrt{E_b}}{\sqrt{N_0}}\right]^2} dx_1$$

$\therefore$ Let $\dfrac{x_1-\sqrt{E_b}}{\sqrt{N_0}} = z$ | When $x_1 = 0$, $z = -\sqrt{\dfrac{E_b}{N_0}}$

$dx_1 = \sqrt{N_0}\, dz$ | $x_1 = -\infty$, $z = \infty$

$$\therefore \quad P_e(1) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-z^2} dz$$

$$\therefore \quad \boxed{P_e(1) = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{N_0}}} \longrightarrow (8)$$

from (7) & (8),

$$P_e(0) = P_e(1) = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{N_0}}$$

If symbol '0' & '1' occurs with equal probability ie. $P(0) = P(1) = 1/2$, then the average probability of symbol error is,

$$P_e = \frac{1}{2}\left[P_e(0) + P_e(1) \right] \qquad P_e = P(0)P_e(0) + P(1)P_e(1)$$

$$\boxed{P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{N_0}}}$$

$\therefore$ For a given channel, as signal energy $\sqrt{E_b}$ increases, avg. probability of error 'P_e' decreases.

* Frequency shift keying techniques using coherent detection:-

* Binary frequency shift keying [BFSK]:- RFID stds, wireless LANN

→ In BFSK, let $s_1(t)$ & $s_2(t)$ be the signal which represents symbol '1' & '0' respectively. This two signals are sinusoidal having two distinct frequencies.

Symbol 1: $\quad s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_1 t \qquad 0 \le t \le T_b \longrightarrow (1)$

Symbol 0: $\quad s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_2 t \qquad 0 \le t \le T_b \longrightarrow (2)$

$f_1 = \frac{n_1}{T_b}$ & $f_2 = \frac{n_2}{T_b}$: where, $n_1, n_2 \rightarrow$ any integer

→ ∴ The orthogonal basis function of unit energy is.

$$\phi_1(t) = \begin{cases} \sqrt{2/T_b} \cos 2\pi f_1 t & 0 \le t \le T_b \\ 0 & \text{elsewhere} \end{cases}$$

$$\phi_2(t) = \begin{cases} \sqrt{2/T_b} \cos 2\pi f_2 t & 0 \le t \le T_b \\ 0 & \text{elsewhere} \end{cases}$$

Expressing (1) & (2) interms of $\phi_1(t)$ & $\phi_2(t)$,

$$s_1(t) = \sqrt{E_b}\ \phi_1(t) \rightarrow (3)$$
$$s_2(t) = \sqrt{E_b}\ \phi_2(t) \rightarrow (4)$$

∴ From this eqns it is clear that the signal space is of two dimension ∴ two msg. points are present & are represented by signal vectors 's_1' & 's_2'.

the co-efficients of $s_1(t)$ are s_{11} & s_{12}

$$\therefore\ s_{11} = \int_0^{T_b} s_1\, \phi_1(t)\, dt = \sqrt{E_b} \rightarrow (5)$$

$$s_{12} = \int_0^{T_b} s_1\, \phi_2(t)\, dt = 0 \longrightarrow (6)$$

s_{12} is zero ∵ $s_1(t)$ & $\phi_2(t)$ are orthogonal

III'ly S_2 co-efficients are S_{21} & S_{22},

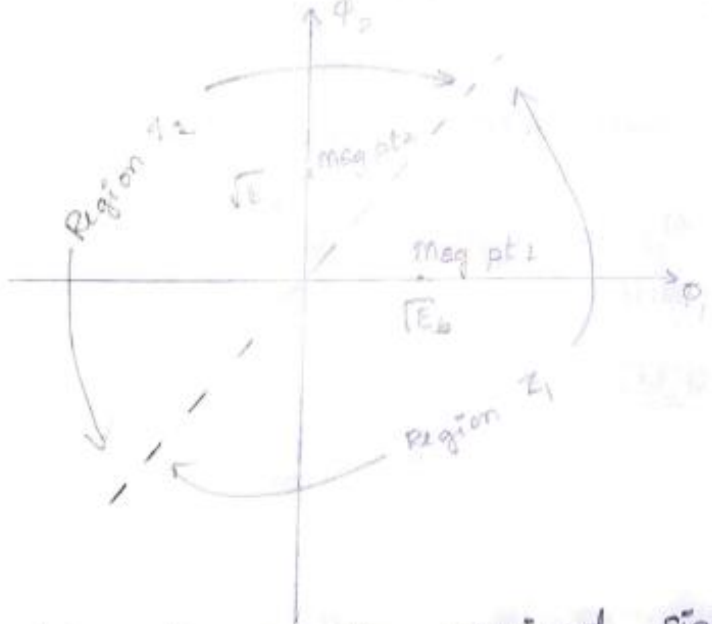
$$S_{21} = \int_0^{T_b} S_2\, \phi_1(t)dt = 0 \quad \because S_2 \text{ \& } \phi_1(t) \text{ are orthogonal}$$

$$S_{22} = \int_0^{T_b} S_2\, \phi_2(t)dt = \sqrt{E_b}$$

$$\therefore S_2 = \begin{bmatrix} S_{21} \\ S_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$ & the co-ordinates are $(0, \sqrt{E_b})$

→ The distance b/w 2 msg points are $\sqrt{2E_b}$

→ constellation diagram is as shown below,



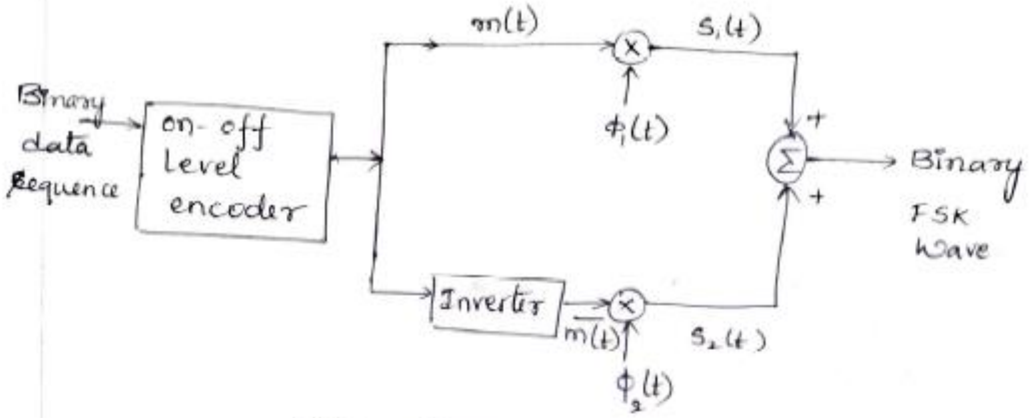
→ Let $x(t)$ be the received signal i.e $x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

where, $x_1 = \int_0^{T_b} x(t)\, \phi_1(t)\, dt$ ↳ (1) & $x_2 = \int_0^{T_b} x(t)\, \phi_2(t) dt$ ↳ (2)

when symbol '1' is transmitted, $x(t) = s_1(t) + w(t)$

when symbol '0' is transmitted, $x(t) = s_2(t) + w(t)$

* Generation :-



BFSK transmitter

→ Binary data is divided into two signals, the first signal is multiplied with $\phi_1(t)$ & the resultant is,

$$s_1(t) = m(t)\,\phi_1(t) \rightarrow (9)$$

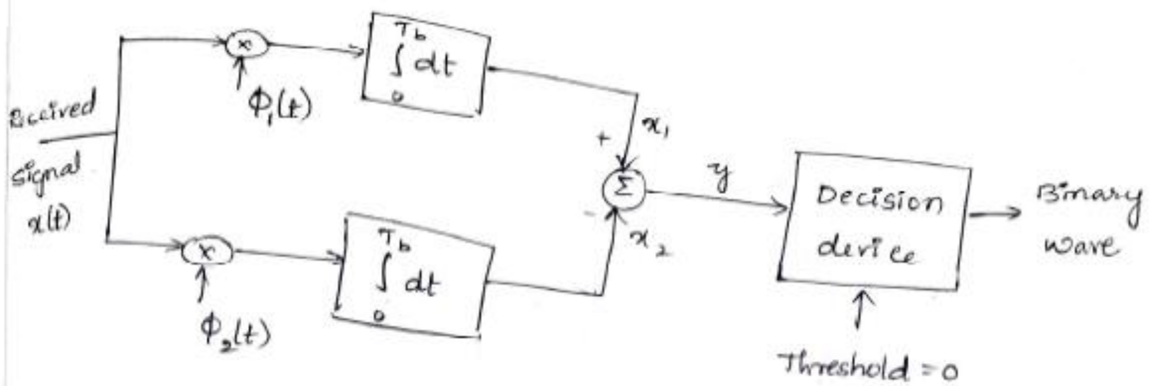
→ ||ly, the second signal is inverted & multiplied with $\phi_2(t)$ & the resultant is,

$$s_2(t) = \overline{m(t)}\,\phi_2(t) \rightarrow (10)$$

→ By using inverter in lower path, when symbol is '1' the oscillating freq. f_1 is switched 'ON' & 'f_2' is 'OFF'. Thus the o/p of summer is a sine wave of frequency 'f_1'.

→ ||ly when symbol is 0, f_2 is ON & f_1 is OFF, thus the o/p of summer is a sine wave of frequency 'f_2'

* Coherent BFSK Receiver :-



→ $x(t)$ is the received signal which is cross correlated with $\phi_1(t)$ & $\phi_2(t)$ to obtain x_1 & x_2. The difference of x_1 & x_2 is 'y' which is fed into decision device.

→ If $y > 0$ ⟶ symbol '1'

If $y < 0$ ⟶ Symbol '0'

* Probability of error calculation :-

→ If '$x(t)$' is received signal, then

$$x(t) = \begin{cases} s_1(t) + w(t) & \text{for symbol '1'} \\ s_2(t) + w(t) & \text{for symbol '0'} \end{cases} \rightarrow (1)$$

Symbol '0' :

i) O/p of upper path is, $x_1(t) = \int_0^{T_b} x(t)\, \phi_1(t)\, dt$

$$x_1(t) = \int_0^{T_b} [s_2(t) + w(t)]\, \phi_1(t)\, dt = s_{21} + w_1$$

But $s_{21} = 0$

Mean of x_1, $E[x_1] = E[W_1] = 0$.

Variance, $Var[x_1] = Var[W_1] = \frac{N_0}{2}$

ii) O/P of lower path is, $x_2 = \int_0^{T_b} x(t)\phi_2(t)\,dt$

$$x_2 = \int_0^{T_b} [s_2(t) + w(t)]\phi_2(t)\,dt = s_{22} + W_2$$

$$x_2 = \sqrt{E_b} + W_2 \longrightarrow (13)$$

Mean, $E[x_2] = E[\sqrt{E_b}] + E[W_2] = 0 + \sqrt{E_b} = \sqrt{E_b}$

Variance, $Var[x_2] = 0 + N_0/2 = \frac{N_0}{2}$

∴ conditional Mean of y, $E[y] = E[x_1] - E[x_2] = 0 - \sqrt{E_b}$

$\hookrightarrow = -\sqrt{E_b}$

∴ $Var[y] = Var[x_1] + Var[x_2]$ ∵ Variance of random variable 'y' is sum of Variance of random variables 'x_1' & 'x_2'

$= \frac{N_0}{2} + \frac{N_0}{2}$

$\hookrightarrow = N_0$

⇒ Conditional PDF when '0' is transmitted,

$$f_y(y|0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$$\hookrightarrow = \frac{1}{\sqrt{2\pi N_0}} e^{-\frac{(y+\sqrt{E_b})^2}{2N_0}}$$

If the γx^r makes wrong decision, ∴ the probability

$$P_e(0) = \int_0 f_y(y|0)\,dy = \int_0^{\infty} \frac{1}{\sqrt{2\pi N_0}} e^{-\left[\frac{y+\sqrt{E_b}}{\sqrt{2N_0}}\right]^2} dy \rightarrow (14)$$

Let $\frac{y+\sqrt{E_b}}{\sqrt{2N_0}} = z$

$\therefore dy = \sqrt{2N_0}\,dz$

When $y = 0$, $z = \sqrt{\frac{E_b}{2N_0}}$

$y = \infty$, $z = \infty$

$$\therefore P_e(0) = \int_{\sqrt{\frac{E_b}{2N_0}}}^{\infty} \frac{1}{\sqrt{2\pi N_0}} e^{-z^2} \sqrt{2N_0}\,dz = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{2N_0}}}^{\infty} e^{-z^2} dz$$

$$\therefore P_e(0) = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \rightarrow (15)$$

$$\text{Ily} \quad P_e(1) = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \rightarrow (16)$$

If symbols '1' & '0' are of equal probability then the avg. probability of error,

$P(0) = P(1) = 1/2$, $\therefore$

$$P_e = P(0)\,P_e(0) + P(1)\,P_e(1)$$

$$= \frac{1}{2}\left[\frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{2N_0}} + \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{2N_0}}\right]$$

$$\therefore \boxed{P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{2N_0}}}$$

Distance b/w two msg. points in FSK is $\sqrt{2E_b}$ & in PSK is $2\sqrt{E_b}$

Larger the distance, smaller the average probability

Q1 A binary data is transmitted at a rate of 10^6 bits/sec over a microwave binary link, assuming channel noise is AWGN with '0' mean & PSD at receiver end is 10^{-10} W/Hz. Find avg power required to maintain an avg. probability of error $\le 10^{-4}$ for a coherent BFSK & determine min BW required.

→ Soln:- bit rate, $R_b = 10^6$ bits/sec, $P_e \le 10^{-4}$, $\frac{N_0}{2} = 10^{-10}$, $N_0 = 2\times10^{-10}$

∴ For BFSK, $P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)$

$$2P_e = \mathrm{erfc}\sqrt{\frac{E_b}{2N_0}} \;;\; 2\times10^{-4} = \mathrm{erfc}\left[\sqrt{\frac{E_b}{4\times10^{-10}}}\right] \rightarrow (1)$$

But, $\mathrm{erfc}(u) = 1 - \mathrm{erf}(u) \rightarrow (2)$

from (1) & (2), $2\times10^{-4} = 1 - \mathrm{erf}\left[\sqrt{\frac{E_b}{4\times10^{-10}}}\right]$

$$\therefore \mathrm{erf}\left[\sqrt{\frac{E_b}{4\times10^{-10}}}\right] = 1 - 2\times10^{-4} = 0.9998 \;;\; \sqrt{\frac{E_b}{4\times10^{-10}}} = \mathrm{erf}^{-1}(0.99$$

$$\sqrt{\frac{E_b}{4\times10^{-10}}} = 2.7$$

$$\frac{E_b}{4\times10^{-10}} = 7.29 \;;\; \Rightarrow \boxed{E_b = 2.916\times10^{-9}\,J}$$

$\therefore E_b = PT_b$

But $T_b = \frac{1}{R_b} = \frac{1}{10^6} = 10^{-6}$ sec

$$\therefore \boxed{P = \frac{E_b}{T_b} = 2.916\times10^{-3}\,W}$$

82. An FSK s/m has a binary data at a rate of 10^6 bits/sec assuming a channel noise is AWGN with '0' mean & PSD = 2×10^{-20} w/Hz. Determine the avg probability of error. Assume coherent detection & amplitude of received sinusoidal signal for both '1' & '0' is 1.2μV

→ $\frac{N_0}{2} = 2\times10^{-20}$, $R_b = 10^6$, $A_m = 1.2\times10^{-6}$

$\therefore E = PT_b = P/R_b$

But, $P = \frac{A_m^2}{2} = \frac{(1.2\times10^{-6})^2}{2} = 7.2\times10^{-13}$ w

$$E_b = \frac{7.2\times10^{-13}}{10^6} = 7.2\times10^{-19} J$$

$$P_e = \frac{1}{2}\, erfc\sqrt{\frac{E_b}{2N_0}} = \frac{1}{2}\, erfc\sqrt{\frac{7.2\times10^{-19}}{8\times\cdot}} = \frac{1}{2}\, erfc(3)$$

$$= \frac{1}{2}\left[1 - erf(3)\right] = \frac{1}{2}\left[1 - 0.9998\right]$$

$$= \frac{1}{2}\times 2\times10^{-4}$$

$\Rightarrow = 10^{-4}$

(1)

Quadriphase - shift keying.

As with binary PSK, information about the message symbols in QPSK is contained in the carrier phase.

* In particular carrier takes on one of four equally spaced values such as $\pi/4$, $3\pi/4$, $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$. For this set of values, we may define the transmitted signal as,

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right], & \begin{cases} 0 \le t \le T_b \\ i = 1, 2, 3, 4. \end{cases} \\ 0, & \text{elsewhere.} \end{cases} \quad \text{—(1)}$$

Goals of Digital comm system are:

1. Low probability of error
2. Efficient utilisation of channel Bandwidth.

'QPSK' is a bandwidth conserving scheme.

where, E_b - is the transmitted signal energy per symbol

and T_b - is the symbol duration.

- The carrier frequency $f_c = \frac{n_c}{T_b}$; n_c - fixed integer.

* Each possible value of phase corresponds to a unique dibit (i.e. pair of bits).

Thus, for example, we may choose Gray encoded set of dibits, 10, 00, 01, and 11, where only a single bit is changed from one dibit to the next.

Signal space diagram of QPSK signals. (constellation diagram) ②

For QPSK, the transmitted signal can be given as,

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right] & \begin{cases} 0 \le t \le T_b \\ i = 1,2,3,4. \end{cases} \\ 0, \quad \text{— ①} & \text{elsewhere.} \end{cases}$$

using trignometric identity,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B. \quad \text{— ②}$$

we can write eqn ① as,

$$s_i(t) = \sqrt{\frac{2E_b}{T_b}}\left[\cos(2\pi f_c t)\cos(2i-1)\frac{\pi}{4} - \sin(2\pi f_c t)\sin\left((2i-1)\frac{\pi}{4}\right)\right] \quad \text{— ③}$$

where, $i = 1,2,3,4$. Based on this representation, we make two observations:

1. There are two orthonormal basis functions, defined by a pair of quadrature carriers:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t), \quad 0 \le t \le T_b. \quad \text{— ④}$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \sin(2\pi f_c t), \quad 0 \le t \le T_b. \quad \text{— ⑤}$$

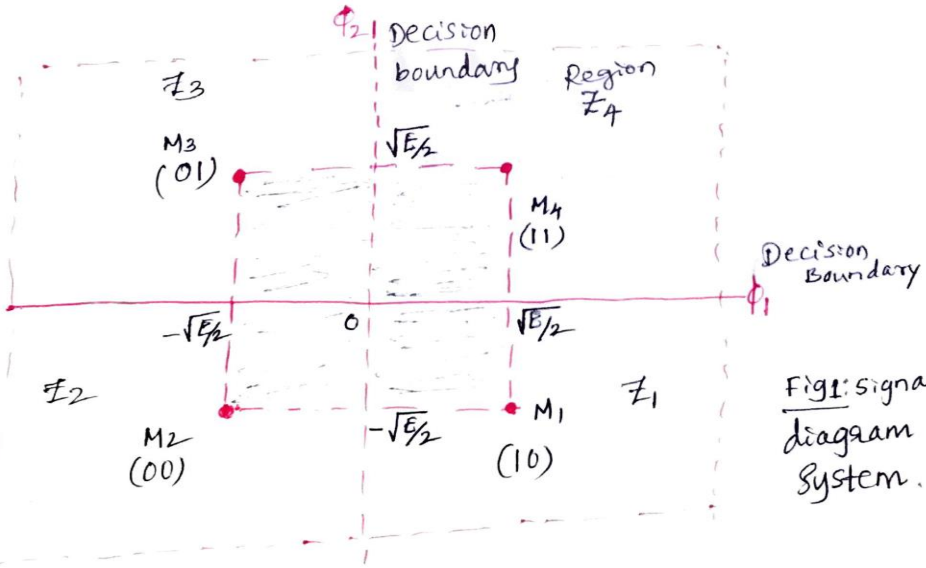


Fig1: signal-space diagram of QPSK System.

2. There are four message points, defined by the two (3) dimensional signal vector.

$$s_i = \begin{bmatrix} \sqrt{E_b}\cos[(2i-1)\pi/4] \\ \sqrt{E_b}\sin[(2i-1)\pi/4] \end{bmatrix}, \quad i = 1, 2, 3, 4.$$

The values of signal vectors s_{i1} and s_{i2} are summarized in below table 1.

Accordingly, a QPSK signal has a two dimensional signal constellation (ie, $N=2$) and four message points (ie. $M=4$) whose phase angles increase in counter clockwise direction, as illustrated in fig1.

Table 1: Signal space characterization of QPSK.

Gray-encoded input dibit	Phase of QPSK signal (radians)	Coordinates of message points s_{i1}	s_{i2}
11	$\pi/4 = 45°$	$+\sqrt{E/2}$	$+\sqrt{E/2}$
01	$3\pi/4 = 135°$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
00	$\frac{5\pi}{4} = 225°$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
10	$\frac{7\pi}{4} = 315°$	$+\sqrt{E/2}$	$-\sqrt{E/2}$

4: Generation and coherent Detection of QPSK Signals. (4)

A block diagram of QPSK transmitter/generator is shown in fig 2(a).

* A distinguishing feature of the QPSK transmitter is the block labeled demultiplexer. The function of the demultiplexer is to divide the binary wave produced by the polar NRZ-level encoder into two seperate binary waves.

* one binary wave represents odd numbered dibits and other represents even numbered dibits. Accordingly, we can make the following statement.

* The QPSK transmitter may be viewed as two binary PSK generators that work in parallel, each at a bit rate equal to one-half the bit rate of the original binary sequence at the QPSK transmitter input.

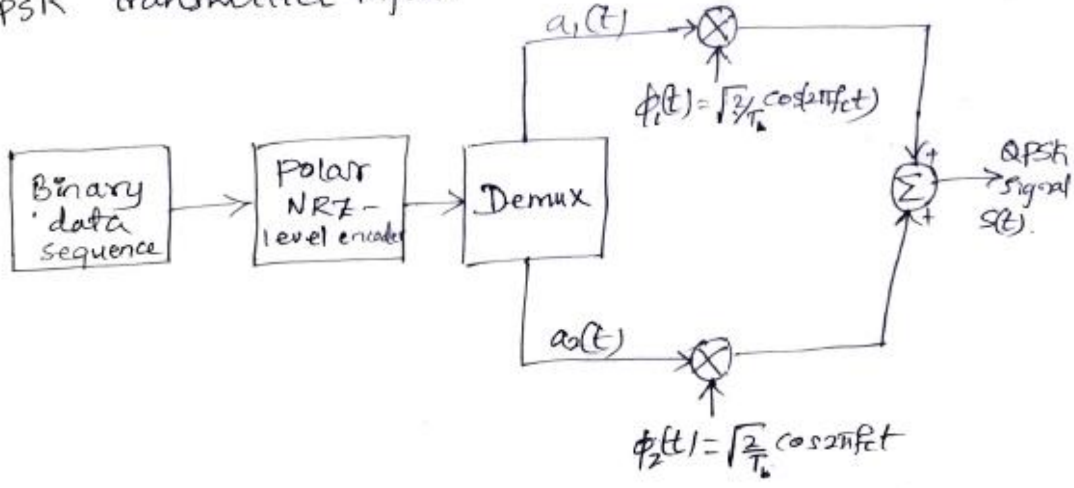


fig 2a: QPSK transmitter.

Fig 2(b) shows the QPSK receiver.

* It can be observed that QPSK receiver is structured in the form of an in-phase path and quadrature path working in parallel.

* The functional composition of the QPSK receiver is as follows:

(1) Pair of correlators, which have common input $x(t)$. The two correlators are supplied with a pair of locally generated orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$, which means that the receiver is synchronized with the transmitter. The correlator outputs are x_1 and x_2.

(2) Pair of decision devices, which act on the correlator outputs x_1 and x_2 by comparing each one with a zero threshold. For inphase channel,

If, $x_1 > 0$ – decision is symbol '1'

elseif $x_1 < 0$ decision is symbol '0'.

Similar binary decisions are made for the quadrature channel. Finally,

(3) Multiplexer – combines the two binary sequences produced by the pair of decision devices. The resulting binary sequence is the estimate of the original binary sequence transmitted.

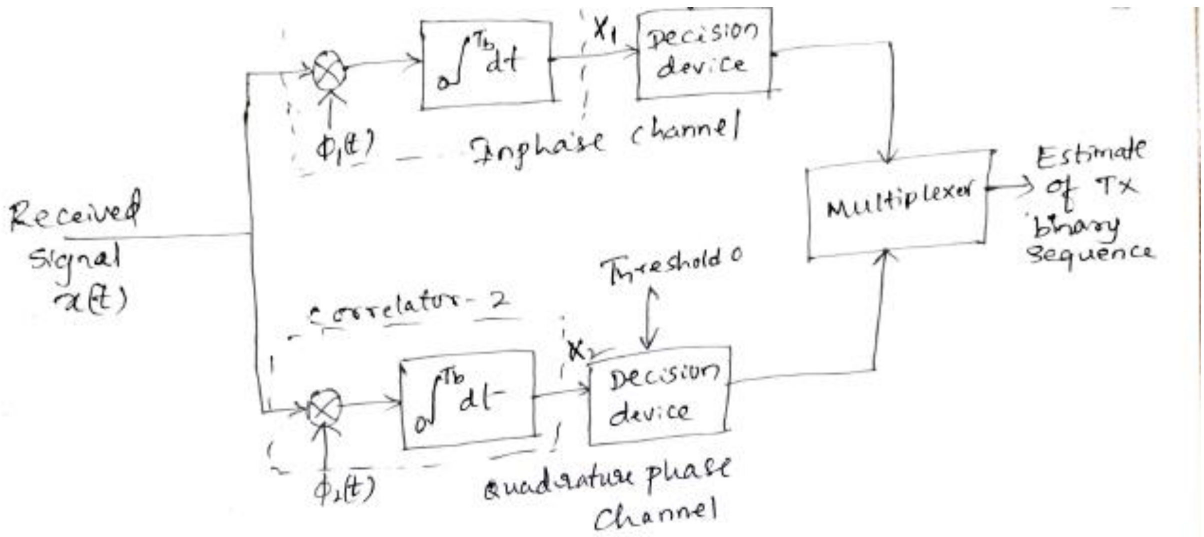


Fig 2(b): Coherent QPSK receiver.

Error Probability of QPSK

In a QPSK system operating on an AWGN channel the received signal $x(t)$ is defined by

$$x(t) = S_i(t) + w(t) \quad \begin{cases} 0 \le t \le T_s \\ i = 1, 2, 3, 4 \end{cases} \quad \text{--- (1)}$$

where $w(t)$ is the sample function of a white Guassian noise process of zero mean and power spectral density of $N_0/2$.

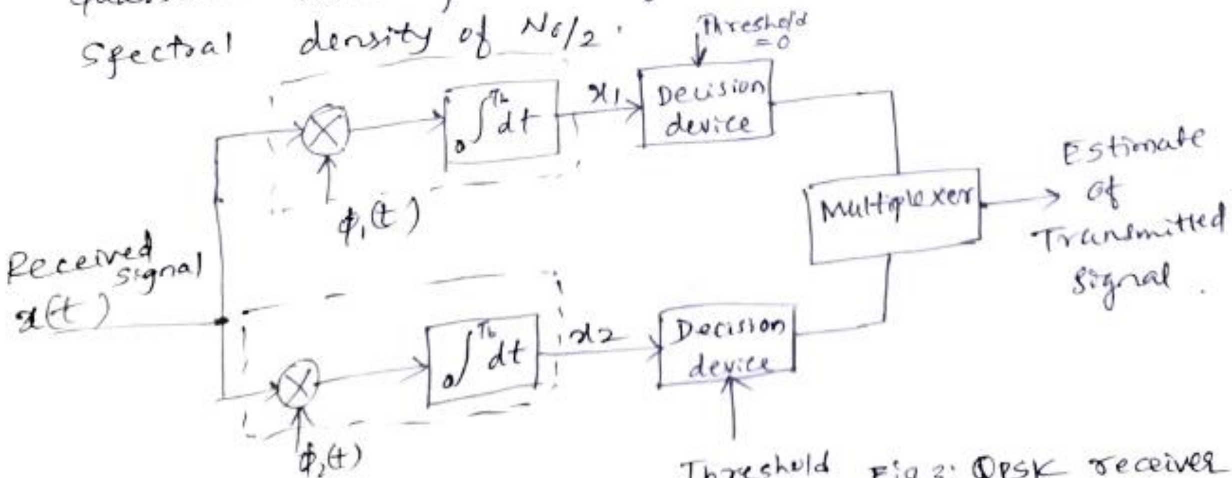


Fig 3: QPSK receiver

Referring to fig③, the two correlator outputs | ⑦
x_1 and x_2 are respectively defined as follows:

$$x_1 = \int_0^{T_b} x(t)\,\phi_1(t)\,dt$$

$$x_1 = \sqrt{E_b}\cos\left[(2i-1)\frac{\pi}{4}\right] + w_1 \quad \text{—②}$$

$$x_1 = \pm\sqrt{\frac{E}{2}} + w_1 \quad \text{—③}$$

[∵ using $i = 1, 2, 3, 4$ in eqn ② we get $\pm\sqrt{E/2}$]

and

$$x_2 = \int_0^{T_b} x(t)\,\phi_2(t)\,dt$$

$$= \sqrt{E_b}\sin\left[(2i-1)\frac{\pi}{4}\right] + w_2 \quad ; \text{ where, } i = 1, 2, 3, 4.$$

$$x_2 = \mp\sqrt{\frac{E}{2}} + w_2 \quad \text{—④}$$

The decision rule is now simply to say that $s_1(t)$ was transmitted if the received signal point associated with the observation vector 'x' falls inside region Z_1; say that $S_2(t)$ is transmitted if the observation vector falls inside region 'Z_2' and so on for other two regions Z_3 & Z_4.

* To calulate the average probability of symbol error, we recall that, the inphase channel 'x_1' and the quadrature phase channel output 'x_2' may be viewed as the individual outputs of two binary PSK receivers. Thus, according to eqn ③ & ④

these PSK receiver are characterized as follows:

* Signal energy per bit equal to $E/2$ and
* noise spectral density equal to $\frac{N_0}{2}$

Hence, using eq2 of PSK, for the average probability of bit error of a coherent binary receiver, we may express the average probability of bit error in the inphase and quadrature paths of the coherent QPSK receiver as

$$p' = Q\left(\sqrt{\frac{E}{N_0}}\right) \cancel{= Q\sqrt{\frac{2E_b}{}}} \quad — (5)$$

where, $Q\left(\frac{E}{N_0}\right) = \frac{1}{2} \operatorname{erfc}\sqrt{\frac{E}{2N_0}}$ & $E = 2E_b$.

The average probability of a correct detection resulting from the combined action of two paths working together is,

$$P_c = (1-p')^2 = \left[1 - Q\left(\sqrt{\frac{E}{N_0}}\right)\right]^2 \quad — (6)$$

$$P_c = 1 - 2Q\left(\sqrt{\frac{E}{N_0}}\right) + Q^2\left(\sqrt{\frac{E}{N_0}}\right) \quad — (7)$$

The average probability of error for QPSK is therefore

$$P_e = 1 - P_c$$

$$P_e = 2Q\left(\sqrt{\frac{E}{N_0}}\right) - Q^2\left(\sqrt{\frac{E}{N_0}}\right) \quad — (8)$$

In the region where $(E/N_0) >> 1$, we may ignore the quadratic term of eq2 (8) and re-write eq2 (8) as,

$$P_e \simeq 2Q\left(\frac{E}{N_0}\right) \quad \text{or} \quad \boxed{P_e \simeq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)} \quad — (9)$$

interms of error fn $\boxed{P_e = \operatorname{erfc}\sqrt{\frac{E_b}{N_0}}} \quad — (10)$

Also,

Bit error rate, $BER = \left(\frac{M/2}{M-1}\right) P_e$.

where M=4 - symbols for QPSK. i.e for large 'M' bit error rate is limited to $\frac{1}{2} P_e$

$\Rightarrow$ $BER = \frac{1}{2} P_e$

$$\boxed{BER = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

Thus, for the same E_b/N_0 and, therefore the same average probability of bit ~~rate~~ error, QPSK system transmits information at twice the bit rate of a binary PSK system for the same channel BW.

(10)

M-ary PSK

QPSK is a special case of ~~M-ary~~ the generic form of PSK commonly referred ~~to~~ as M-ary PSK. ~~where~~

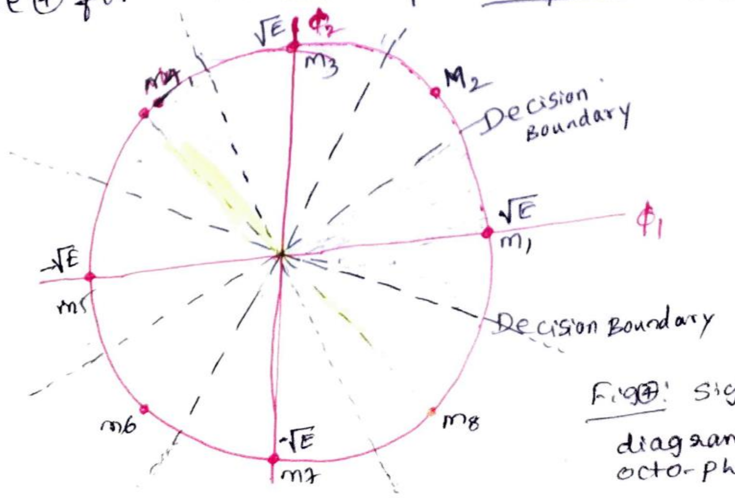
* In M-ary PSK the phase of the carrier takes on one of M-possible values: $\theta_i = 2(i-1)\frac{\pi}{M}$; $i = 1, 2, \ldots M$.
Accordingly, during each signalling interval of duration T, one of the M possible signals

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + \frac{2\pi}{M}(i-1)\right], \quad i = 1, 2, \ldots M \qquad (1)$$

is sent, where E - is the signal Energy/symbol
T - symbol interval
f_c - carrier freq $= \frac{n_c}{T}$; n_c - fixed integer.

* Each $S_i(t)$ may be expanded interms of the same two basis functions $\phi_1(t)$ and $\phi_2(t)$; the signal constellation of M-ary PSK is, therefore two dimensional.

* The M message points are equally spaced on a circle of radius $\sqrt{E}$ and center at the origin, as illustrated in figure (4) for the case of octophase shift keying (ie M = 8).



(M = 8).

Fig(4): signal space diagram for octo-phase keying

* Suppose that the transmitted signal corresponds to the message point m_1, suppose that E/N_0 is large enough to consider the nearest two points 'm_2' and 'm_8' as potential candidates for being mistaken for 'm_1' due to channel noise, the euclidean distance for each of these two points from m_1 is (for $M=8$)

$$d_{12} = d_{18} = 2\sqrt{E}\sin\frac{\pi}{M} \quad \text{—(2)}$$

Hence, using concept of union bound and Q-function average probability of symbol error for coherent M-ary psk is

$$P_e \simeq 2Q\left[\sqrt{\frac{2E}{N_0}}\sin\left(\frac{\pi}{M}\right)\right] \quad \text{—(3)}$$

where, $M \geq 4$.

if $M=4$, P_e - will be error probability of QPSK.

* Channel Bandwidth for M-ary PSK

The channel Bandwidth required to pass M-ary PSK signals through an analog channel as,

$$B = \frac{2}{T} \quad \text{(1)}; \text{ where } T \text{ is the symbol duration.}$$

But symbol duration 'T' for M-ary psk is defined by

$$T = T_b \log_2(M) \quad \text{(2)}; \quad T_b \text{ - bit duration.}$$

also, interms of bit rate, $R_b = \frac{1}{T_b}$, we can write eqn (1) as,

$$\boxed{B = \frac{2}{T_b \log_2 M} = \frac{2R_b}{\log_2 M}} \quad \text{bits/sec.}$$

(12)

* Bandwidth Efficiency (ρ)

we know that, for M-ary PSK, channel BW is given by,

$$B = \frac{2R_B}{\log_2 M} \quad -①$$

using eqn ① BW efficiency is given by

$$\rho = \frac{R_B}{B}$$

$$\Rightarrow \boxed{\rho = \frac{\log_2 M}{2}} \quad \text{bits/sec/Hz}$$

Table shown below indicates BW efficiency of M-ary PSK signals for different M-value

M	2	4	8	16	32	64
ρ bits/s/Hz	0.5	1	1.5	2	2.5	3

Based on above data we can say that,

"As the number of states of M-ary PSK is increased, the BW efficiency is improved at the expense of error performance"

Also, If we want to improve error probability then E_b/N_0 must be increased to compromise for increase in 'M'.

* M-ary Quadrature Amplitude Modulation (QAM)

The QAM is a hybrid form of modulation, in that the carrier experiences amplitude as well as phase modulations.

(13)

* The transmitted M-ary QAM signal for symbol 'k' can be defined interms of E_0 as,

$$s_k(t) = \sqrt{\frac{2E_0}{T}}\, a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}}\, b_k \sin(2\pi f_c t) \begin{cases} 0 \le t \le T \\ k = 0, \pm 1, \pm 2 \ldots \end{cases} \quad \text{—(1)}$$

The signal $s_k(t)$ involves two-phase quadrature carriers, each one of which is modulated by a set of discrete amplitudes; hence the terminology "quadrature amplitude modulation". [E_0 - is the energy of message signal with the lowest amplitude]

The two orthogonal basis functions are:

$$\left.\begin{aligned} \phi_1(t) &= \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \qquad 0 \le t \le T \\ \phi_2(t) &= \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \qquad 0 \le t \le T \end{aligned}\right\} \quad \text{—(2)}$$

* QAM square constellations.

A QAM square constellation can be factored into the product of the corresponding L-ary PAM constellation with itself.

* In figure(5), we have constructed two signal constellations for 4-ary PAM, one vertically oriented along ϕ_2-axis in part a of the figure, and the other horizontaly oriented along ϕ_1-axis in part b. These two parts are spatially orthogonal.

* In developing two dimensional structure of M-ary QAM the following points need to be remembered.

(1) The same binary sequence is used for both 4-ary PAM constellations.

(14)

(2) Gray coding rule is used

(3) we move from one quadrant to the next in a counter clock wise direction.

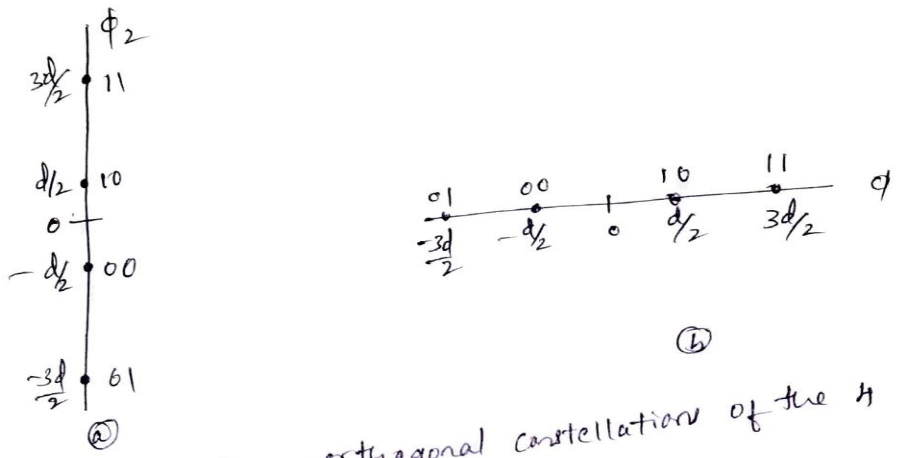


Fig 5 : The two orthogonal constellation of the 4 ary PAM.

Step 1 : First quadrant constellation

we use the codewords along the positive parts of the ϕ_1 and ϕ_2 - axis.

$$\underset{\text{Top to bottom}}{\begin{bmatrix} 11 \\ 10 \end{bmatrix}} \underset{\text{left to right}}{\begin{bmatrix} 10 & 11 \end{bmatrix}} \rightarrow \begin{bmatrix} 1110 & 1111 \\ 1010 & 1011 \end{bmatrix} \rightarrow \text{First quadrant codes}$$

Step 2 : Second quadrant constellation

$$\underset{\text{top to bottom}}{\begin{bmatrix} 11 \\ 10 \end{bmatrix}} \quad \underset{\text{left to right}}{\begin{bmatrix} 01 & 00 \end{bmatrix}} \rightarrow \underset{\text{Second quadrant}}{\begin{bmatrix} 1101 & 1100 \\ 1001 & 1000 \end{bmatrix}}$$

(15)

Step3: Third quadrant constellation.

$$\begin{bmatrix} 00 \\ 01 \end{bmatrix} \quad [01 \quad 00] \rightarrow \begin{bmatrix} 0001 & 0000 \\ 0101 & 0100 \end{bmatrix}$$

top to bottom — left to right — Third quadrant

Step4: Fourth quadrant constellation.

$$\begin{bmatrix} 00 \\ 01 \end{bmatrix} \quad [10 \quad 11] \rightarrow \begin{bmatrix} 0010 & 0011 \\ 0110 & 0111 \end{bmatrix}$$

Top to bottom — left to right — Fourth quadrant

The final step is to piece together these four constituent 4-ary PAM to construct M-ary QAM (M=16) as described in fig 6

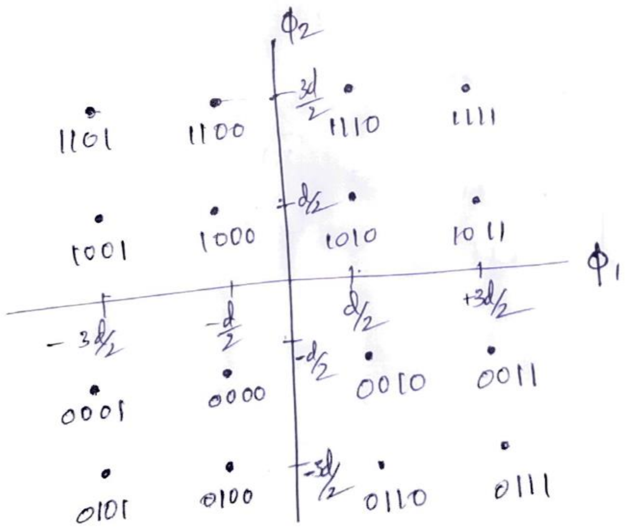
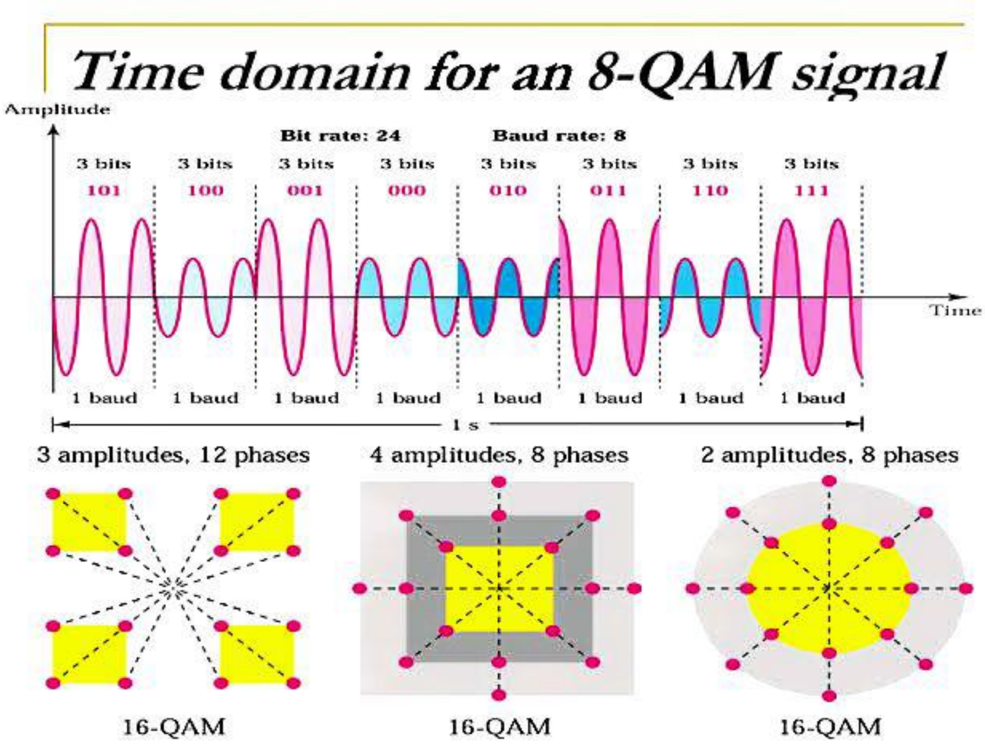


Fig 6: Signal space diagram for M-ary QAM for M=16.

Average error probability is, $P_e \approx 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3E_{av}}{(M-1)N_0}}\right]$


Time domain for an 8-QAM signal
Amplitude
Bit rate: 24
Baud rate: 8
3 bits 3 bits 3 bits 3 bits 3 bits 3 bits 3 bits 3 bits
101 100 001 000 010 011 110 111
Time
1 baud 1 baud 1 baud 1 baud 1 baud 1 baud 1 baud 1 baud
1 s
3 amplitudes, 12 phases
4 amplitudes, 8 phases
2 amplitudes, 8 phases
16-QAM
16-QAM
16-QAM

* Non-coherent binary modulation techniques:-

* Non coherent orthogonal modulation:-

→ Consider two orthogonal signals $s_1(t)$ & $s_2(t)$ which have equal energy. During the interval $0 \le t \le T$ one of these two signals is sent over an imperfect channel that shifts the carrier phase by an unknown amount.

→ Let $g_1(t)$ & $g_2(t)$ be the phase shifted version of $s_1(t)$ & $s_2(t)$ & these two signals $g_1(t)$ & $g_2(t)$ are orthogonal & have equal energy such scheme is called non-coherent orthogonal modulation.

→ The channel adds white gaussian noise $w(t)$ of zero mean &

∴ The received signal $x(t)$ is,

$$x(t) = \begin{cases} g_1(t) + w(t) & 0 \le t \le T \\ g_2(t) + w(t) & 0 \le t \le T \end{cases} \rightarrow (1)$$

$x(t)$ is used to discriminate b/w $s_1(t)$ & $s_2(t)$

→ Fig.(a) depicts a receiver, it consists of a pair of matched filter with basis function $\phi_1(t)$ & $\phi_2(t)$

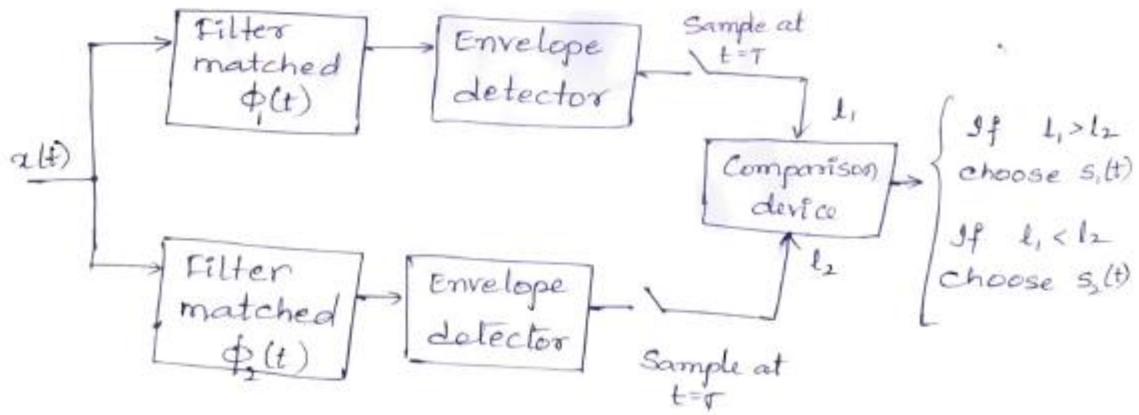


fig 1: Generalized binary Rxr for noncoherent orthogonal modulation

→ The matched filter o/p's are envelope detected, sampled & then compared with each other

→ If $l_1 > l_2$, decision is in favor of $s_1(t)$

If $l_1 < l_2$, decision is in favor of $s_2(t)$

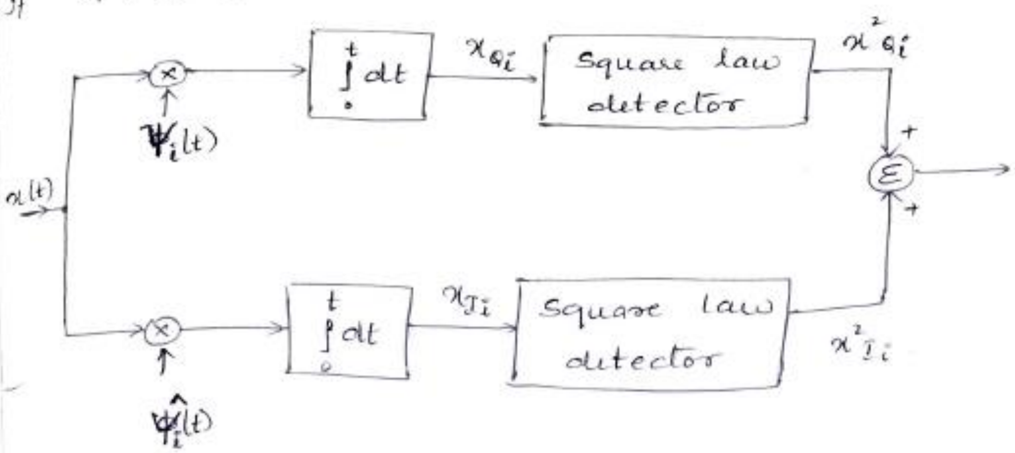


fig 2: Quadrature receiver equivalent

→ Quadrature receiver equivalent of non-coherent Rx is shown in fig.

→ Upper path is called in-phase path, the signal $x(t)$ is correlated with $\psi_i(t)$ representing a scaled version of the transmitted signal $s_1(t)$ & $s_2(t)$ with zero carrier phase.

→ Lower path is called quadrature path, $x(t)$ is correlated with $\hat{\psi}_i(t)$ representing 90° phase shifted version of $\psi_i(t)$ ∴ $\psi_i(t)$ & $\hat{\psi}_i(t)$ are orthogonal to each other and there by demodulate the received signal when passed through integrator and square law device.

Non coherent BFSK :-

→ In BFSK, the transmitted signal is defined by,

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_i t & 0 \le t \le T_b \\ 0 & \text{elsewhere} \end{cases}$$

→ Transmission of 'f_1' represents symbol '1'.
Transmission of f_2 represents symbol '0'

→ Non-coherent detection of FSK includes two matched filters

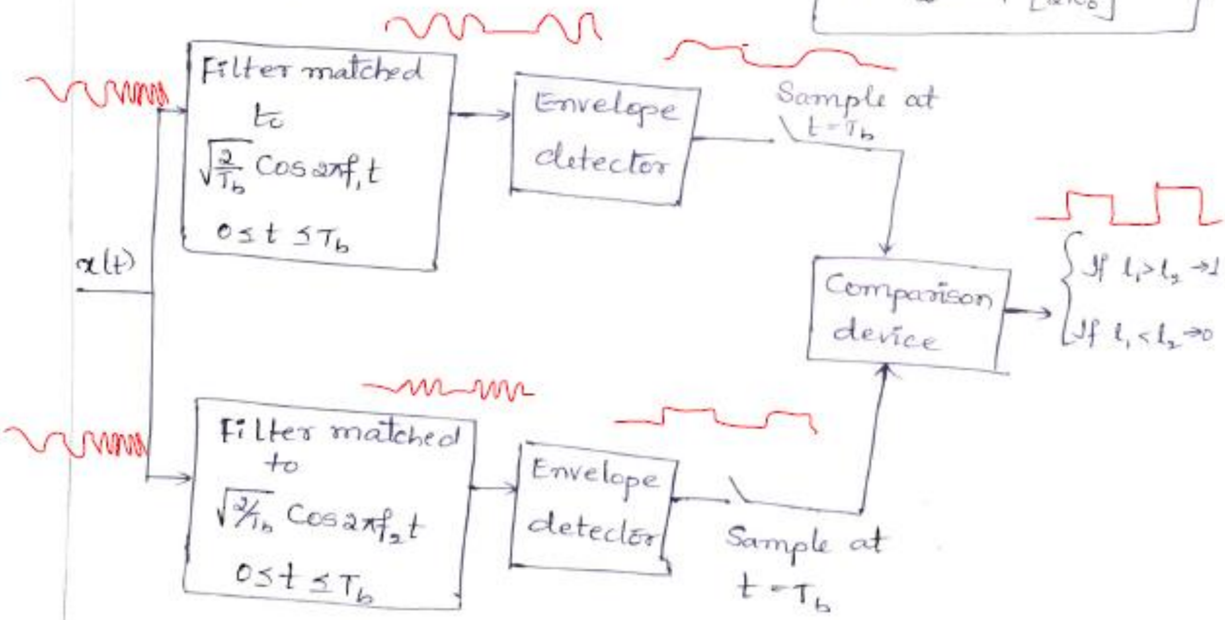
→ The filter in the upper path is matched to $\sqrt{\frac{2}{T_b}} \cos 2\pi f_1 t$ & the filter in the lower path is matched to $\sqrt{\frac{2}{T_b}} \cos 2\pi f_2 t$: $0 \le t \le T_b$.

→ Matched filter o/p is envelope detected & sampled at rate 'T_b' & this values are compared

→ If $l_1 > l_2$:- decision is in favor of symbol '1'
If $l_1 < l_2$:- decision is in favor of symbol '0'

→ The noncoherent BFSK is special case of noncoherent orthogonal modulation with $T = T_b$ & $E = E_b$

→ ∴ The avg. probability of error, $\boxed{P_e = \frac{1}{2} \exp\left[\frac{-E_b}{2N_0}\right]}$



Noncoherent BFSK receiver

Differential phase shift Keying :-

DPSK is noncoherent version of PSK

The two basic operations at transmitted are,

* Differential encoding of i/p binary wave
* Phase shift keying.

To send symbol '0', a phase lead of 180° is added to signal waveform & for symbol '1' phase of waveform is unchanged.

- Receiver is equipped with storage capability, so that it can store relative phase difference b/w two successive bit intervals.
- Suppose the transmitted DPSK signal equals $\sqrt{\frac{E_b}{2T_b}} \cos 2\pi f_c t$ for $0 \le t \le T_b$

Let $s_1(t)$ indicate symbol '1' transmission

$$s_1(t) = \begin{cases} \sqrt{E_b/2T_b} \cos 2\pi f_c t & 0 \le t \le T_b \\ \sqrt{E_b/2T_b} \cos 2\pi f_c t & T_b \le t \le 2T_b \end{cases} \longrightarrow (1)$$

Let $s_2(t)$ indicate symbol '0' transmission.

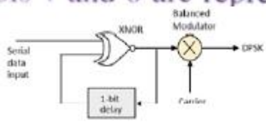
$$s_2(t) = \begin{cases} \sqrt{\frac{E_b}{2T_b}} \cos 2\pi f_c t & 0 \le t \le T_b \\ \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t + \pi) & T_b \le t \le 2T_b \end{cases} \longrightarrow (2)$$

- From eqn (1) & (2), $s_1(t)$ & $s_2(t)$ are indeed orthogonal over two bit interval $0 \le t \le 2T_b$

DPSK is special case of noncoherent orthogonal modulation with $T = 2T_b$ & $E = 2E_b$. the average probability

Illustration of DPSK

Consider the input binary sequence, denoted $\{b_k\}$, to be 10010011, which is used to derive the generation of a DPSK signal. The differentially encoded process starts with the reference bit 1. Let $\{d_k\}$ denote the differentially encoded sequence starting in this manner and $\{d_{k-1}\}$ denote its delayed version by one bit. The complement of the modulo-2 sum of $\{b_k\}$ and $\{d_{k-1}\}$ defines the desired $\{d_k\}$, as illustrated in the top three lines of Table 7.6. In the last line of this table, binary symbols 1 and 0 are represented by phase-shifts of 1 and π radians.



DPSK Modulator

Table 7.6 Illustrating the generation of DPSK signal

	reference								
$\{b_k\}$		1	0	0	1	0	0	1	1
$\{d_{k-1}\}$		1	1	0	1	1	0	1	1
Differentially encoded sequence $\{d_k\}$	1	1	0	1	1	0	1	1	1
Transmitted phase (radians)	0	0	π	0	0	π	0	0	0