Module - 2

Signalling over AWGN channels- Detection & Estimation:

The two fundamental issues in digital commn. are :
- detection
- estimation

} in the presence of additive noise.

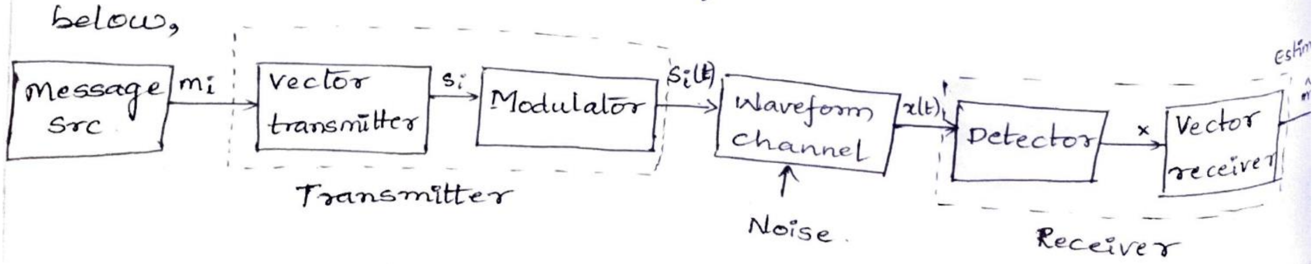
Detection theory deals with the design & evaluation of a decision making processor that observes the received signal & guesses which particular symbol was transmitted according to some set of rules.

Estimation theory deals with the design & evaluation of a processor that uses information in the received signal to extract estimates of physical parameters of interest.

- Results of detection & estimation are always subjected to errors, the challenge is to control the errors so as to ensure an acceptable quality of performance.

Model of digital Communication S/m:-

The conceptual model of DCS is as shown below,



Conceptualized model of a DCS.

→ At the Tx' i/p, the message source emits only one symbol every 'T' seconds, with M symbols which we denote by $m_1, m_2 \dots m_M$

→ We assume that all 'M' symbols of the alphabet are equally likely. Then the priori probability of the message source o/p is,

$$p_i = P(m_i \text{ emitted}) = \frac{1}{M} \quad \text{for all } i$$

→ When the source o/p $m = m_i$, the vector transmitter o/p is,

$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} \quad i = 1, 2 \dots M \quad , \text{ where } N \le M.$$

→ With this vector as i/p, the modulator then constructs a distinct signal $s_i(t)$ of duration 'T' seconds

→ The signal $s_i(t)$ is of finite energy & is given by,

$$E_i = \int_0^T s_i^2(t)\, dt \qquad i = 1, 2 \dots M$$

Where, $s_i(t)$ is real valued.

→ The signal chosen for transmission depends on,

* Incoming message
* The signals transmitted in preceding time slots
* Its characterization depends on the nature of available physical channel for communication

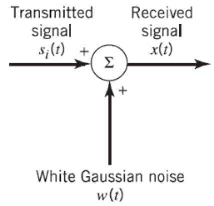
→ The characteristics of the channel are:

* The channel is linear, with a bandwidth sufficient for the transmission of modulator o/p $s_i(t)$ without distortion

* The transmitted signal is disturbed by additive, zero mean, stationary, Gaussian noise denoted by $W(t)$. Such a channel is referred as additive white Gaussian noise [AWGN] channel

The received random process, $x(t)$ is

$$x(t) = s_i(t) + W(t) \qquad 0 \le t \le T, \quad i = 1, 2 \ldots M.$$



AWGN model of a channel.

where, $x(t)$ is represented by sample function $x(t)$
$W(t)$ is represented by sample function $w(t)$

∴ $x(t)$ is referred as received signal.

The estimation of the transmitted signal at the receiver is done in 2 stages,

* Detector operates on received signal to produce a vector of random variable 'x'.
* By using an observation vector 'x', s_i & P_i, the vector receiver produces $\hat{m}$.

To minimize the average probability of symbol error,

$$P_e = \pi_1 P(\hat{m} = 0 | 1 \text{ sent}) + \pi_2 P(\hat{m} = 1 | 0 \text{ sent})$$

Where, π_1 & π_2 → prior probabilities of transmitting symbol '1' or '0'

$P(\hat{m} = 0 | 1 \text{ sent})$ & $P(\hat{m} = 1 | 0 \text{ sent})$ → conditional probabilities

The main motivation for reducing the average probability of error is to make the digital communication system as reliable as possible.

Gram - schmidt orthogonalization procedure :-

→ According to the model in fig a , the task of transforming an incoming msg. m_i, $i = 1, 2 \dots M$ into a modulated wave $s_i(t)$ can be divided into separate discrete time & continuous time operation. The justification for this is given by GSOP.

→ It allows to represent any set of 'M' energy signal, as linear combinations of 'N' orthonormal basis function $[N \leq M]$ ∴ $s_1(t) \dots s_2(t)$ can be represented as,

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\, \phi_j(t) \qquad 0 \leq t \leq T, \quad i = 1, 2 \dots M \qquad \rightarrow (1)$$

→ ∴ The co-efficients are,

$$s_{ij} = \int_0^T s_i(t)\, \phi_j(t)\, dt \qquad i = 1, 2 \dots M, \quad j = 1, 2 \dots N. \qquad \rightarrow (2)$$

→ The basis functions $\phi_1(t) \dots \phi_N(t)$ are orthonormal,

$$\therefore \int_0^T \phi_i(t)\, \phi_j(t) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \qquad \rightarrow (*)$$

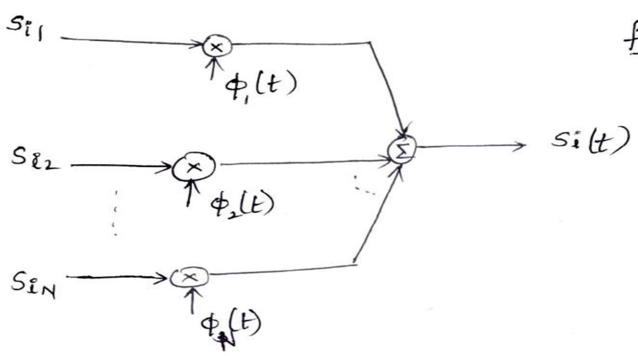
→ The above eqn. states that,

* Each basis function is normalized to have unit energy.
* Basis function, $\phi_1(t) \dots \phi_N(t)$ are orthogonal w.r.t each other over the time $0 \leq t \leq T$.

→ Given the set of co-efficients $\{s_{ij}\}$ $j = 1, 2 \dots N$, operating as i/p , $s_i(t)$ can be generated as shown in fig.(a) with the help of eqn (1).

→ It consists of a bank of N multiplier, with each multiplier supplied with its own basis function, followed by a summer.

→ By using the scheme shown in fig.(b), given the set of signals $\{s_i(t)\}, i = 1, 2 \ldots M$, we can calculate the co-effecient s_{ij}.



fig(a):- Scheme for generating the signals $s_i(t)$

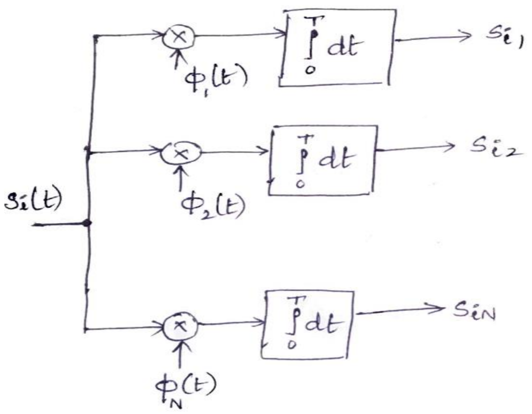


fig.(b) :- Scheme for generating the set of coefficient $\{s_{ij}\}$

→ Gram - schmidt orthogonalization procedure can be proved with 2 stages;

Stage 1:

→ First we need to check whether or not the given set of signals $s_1(t), s_2(t) \ldots s_M(t)$ is linearly independent

→ If not, then there exists a set of co-efficients $a_1, a_2 \ldots a_M$ not all equal to zero such that

$$a_1 s_1(t) + a_2 s_2(t) + \ldots + a_M s_M(t) = 0 \qquad 0 \le t \le T \rightarrow (1)$$

→ If $a_M \neq 0$, then $s_M(t) = -\left[\frac{a_1}{a_M} s_1(t) + \frac{a_2}{a_M} s_2(t) + \ldots + \frac{a_{M-1}}{a_M} s_{M-1}(t)\right] \rightarrow (2)$

→ Now consider next set of signals $s_1(t), s_2(t) \dots s_{M-1}(t)$, check whether this set of signal is linearly independent or not.

→ If not, then there exists a set of numbers $b_1, b_2 \dots b_{M-1}$ not all equal to zero such that

$$b_1 s_1(t) + b_2 s_2(t) + \dots + b_{M-1} s_{M-1}(t) = 0 \quad ; \quad 0 \le t \le T \rightarrow (3)$$

→ Suppose that $b_{M-1} \neq 0$, then $s_{M-1}(t)$ can be expressed as linear combination of the remaining $M-2$ signals,

$$s_{M-1}(t) = -\left[\frac{b_1}{b_{M-1}} s_1(t) + \frac{b_2}{b_{M-1}} s_2(t) + \dots + \frac{b_{M-2}}{b_{M-1}} s_{M-2}(t)\right] \rightarrow (4)$$

→ ∴ The testing of set of signals for linear independence continous, till we get a linear independent subset of the original set of signals.

* → From this we come to know that, each member of the original set of signals $s(t) \dots s_M(t)$ can be expressed as linear combination of this subset of 'N' signals.

Stage 2:- It is possible to construct a set of N orthonormal basis functions $\phi_1(t), \phi_2(t) \dots \phi_N(t)$ from the linear independent signals $s_1(t), s_2(t) \dots s_N(t)$.

→ The first basis function is, $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \rightarrow (1)$

where, $E_1 \rightarrow$ energy of signal $s_1(t)$.

$$\therefore s_1(t) = \sqrt{E_1}\, \phi_1(t) = s_{11} \phi_1(t) \rightarrow (2)$$

where, $s_{11} = \sqrt{E_1}$ & $\phi_1(t)$ has unit energy.

→ The co-efficient s_{21} is, $s_{21} = \int_0^T s_2(t)\, \phi_1(t)\, dt \rightarrow (3)$

→ ∴ The intermediate function is, $g_2(t) = s_2(t) - s_{21}\phi_1(t) \rightarrow (4)$ which is orthogonal to $\phi_1(t)$. $0 \le t \le T_b$

The second basis function is defined as,

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}} \longrightarrow (5)$$

Subs. eqn. (4) in (5),

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}} \longrightarrow (6)$$

$$\hookrightarrow = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

where, $E_2 \rightarrow$ energy of signal, $s_2(t)$.

$\therefore$ From eqn. (5), $\int_0^T \phi_2^2(t)\,dt = 1$

& from (6), $\int_0^T \phi_1(t)\,\phi_2(t)\,dt = 0$

$\therefore$ $\phi_1(t)$ & $\phi_2(t)$ form an orthonormal set.

$\therefore$ In general,
$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\,\phi_j(t)$$

where, $s_{ij}, j = 1, 2 \ldots i-1$

$$s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt$$

$$\therefore \phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,dt}} \quad i = 1, 2 \ldots N.$$

$\therefore$ The derived subset of linearly independent signals $s_1(t), s_2(t) \ldots s_N(t)$ may be expressed as a

→ The co-efficients are defined as,

$$s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt \quad , \quad \begin{matrix} i = 1, 2 \ldots M \\ j = 1, 2 \ldots N. \end{matrix} \quad \rightarrow (2)$$

→ The real-valued basis function $\phi_1(t) \ldots \phi_N(t)$ form an orthonormal set,

$$\text{i.e.} \quad \int_0^T \phi_i(t)\,\phi_j(t)\,dt = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \rightarrow (3)$$

Where, δ_{ij} → kronecker delta

→ From above expression it is clear that, * the first condition states that each basis function is normalized to have unit energy.

* the second condition indicates that the basis functions are orthogonal w.r.t each other over the interval $0 \leq t \leq T$

→ For known value of i, the set of co-efficient $\{s_{ij}\}_{j=1}^{N}$ may be viewed as an N-dimensional signal vector, denoted by s_i

→ The scheme shown in fig.(a) can be used to generate $s_i(t)$. It consists of N multipliers followed by summer. & it is called "synthesizer"

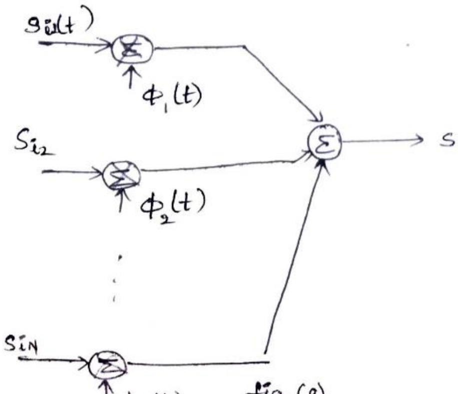


fig.(a)

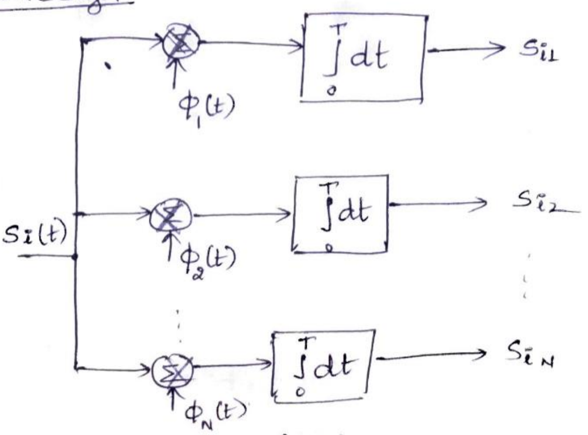


fig.(b)

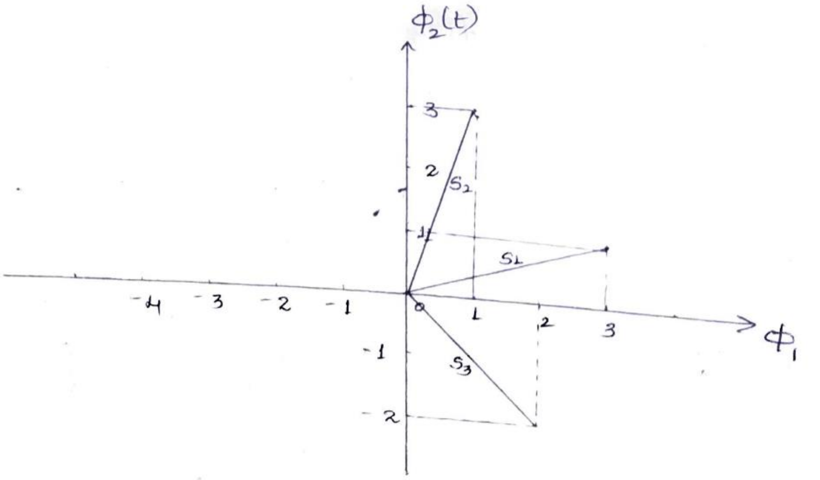
Fig(b). shows the scheme for calculating the co-efficients $s_{i1}, s_{i2} \ldots s_{iN}$ with the help of $s_i(t)$. & basis function It consists of bank of N product integrators or correlators with a common i/p & each one of them supplied with its own basis function It is viewed as an analyzer.

→ Each signal stated in the set $s_i(t)$ is completely determin by the signal vector.

$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} \qquad i = 1, 2 \ldots M.$$

→ Extending two or three dimensional Euclidean space to an N-dimensional Euclidean space, we may visuali the ~~set dim~~ of signal vectors $\{s_i | i = 1, 2 \ldots M\}$ as defining a corresponding set of M points in an N-dimensiona Euclidean space with N mutually perpendicular axes labeled $\phi_1, \phi_2 \ldots \phi_N$. This N-dimensional Euclidean space is called signal space.

→ This provides the mathematical basis for the geometric representation of energy signals as shown in fig.(c).



Fig(c). Geometric representation of signals for $N = 2$ & $M = 3$

→ In an N-dimensional Euclidean space, lengths & angles between vectors are defined.

→ The length of a signal vector s_i is dinoted by $\|s_i\|$

→ The squared length of any signal vector s_i is defined by the inner product or dot product

ie. $\|s_i\|^2 = s_i^T s_i = \sum_{j=1}^{N} s_{ij}^2 \quad i = 1, 2 \cdots M. \quad \rightarrow (4)$

where, s_{ij} = j^{th} element of s_i

T = transposition of a matrix.

→ ∴ Energy of a signal is, $E_i = \int_0^T s_i^2(t)\,dt \rightarrow (5)$

Subs Eqn (1) in (5)

$$E_i = \int_0^T \left[\sum_{j=1}^{N} s_{ij}\,\phi_j(t)\right]\left[\sum_{k=1}^{N} s_{ik}\,\phi_k(t)\right] dt \longrightarrow (6)$$

$$E_i = \sum_{j=1}^{N}\sum_{k=1}^{N} s_{ij}\,s_{ik}\int_0^T \phi_j(t)\,\phi_k(t)\,dt \longrightarrow (7)$$

Subs eqn (3) in (7) & simplifying yields,

$$E_i = \sum_{j=1}^{N} s_{ij}^2$$

From eqn (4), $\sum_{j=1}^{N} s_{ij}^2 = \|s_i\|^2$, ∴ $\boxed{E_i = \|s_i\|^2} \rightarrow (8)$

∴ From eqn (4) & (8), it is clear that energy of a signal is equal to the squared length of the corresponding signal vector s_i.

→ In the case of a pair of signals $s_i(t)$ & $s_k(t)$ represented by the signal vector s_i & s_k, $\int_0^T s_i(t)\,s_k(t)\,dt = s_i^T s_k \rightarrow (9)$

∴ Eqn (9) states that, the inner product of the energy signals $s_i(t)$ & $s_k(t)$ over the interval $[0, T]$ is equal to the inner product of their respective vector representations s_i & s_k.

→ $$\|s_i - s_k\|^2 = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2 = \int_0^T [s_i(t) - s_k(t)]^2 dt$$

where, $\|s_i - s_k\| \rightarrow$ Euclidean distance d_{ik} b/w. the points represented by the signal vector s_i & s_k.

→ Angle θ_{ik} subtended b/w two signal vectors s_i & s_k is.

$$\cos\theta_{ik} = \frac{s_i^T s_k}{\|s_i\| \|s_k\|}$$

Two vectors are orthogonal to each other if their inner product $s_i^T s_k$ is zero.

ie. If $s_i^T s_k = 0$, $\cos\theta_{ik} = \frac{0}{\|s_i\| \|s_k\|} = 0$

$\therefore \theta_{ik} = \cos^{-1} 0$; $\boxed{\theta_{ik} = 90°}$

* Conversion of the continuous AWGN channel into a Vector channel :-

→ Suppose that the i/p to the bank of N product integrators [fig b] is received signal $x(t)$,

$$x(t) = s_i(t) + w(t) \quad \begin{cases} 0 \le t \le T \rightarrow (1) \\ i = 1, 2 \ldots M \end{cases}$$

where, $w(t) \rightarrow$ AWGN with zero mean & PSD $= \frac{N_0}{2}$

→ The o/p of correlator 'j' is the sample value of a random variable 'x_j', whose sample value is defined by,

$$x_j = \int_0^T x(t)\phi_j(t)\, dt = \int_0^T [s_i(t) + w(t)]\phi_j(t)\, dt = s_{ij} + w_j \quad ; j = 1, 2 \ldots N \rightarrow (2)$$

→ 'w_j' is given by, $w_j = \int_0^T w(t)\phi_j(t)\, dt \rightarrow (4)$

& $s_{ij} = \int_0^T s_i(t)\phi_j(t)\, dt \rightarrow (3)$

→ Consider a new stochastic process $x'(t)$ whose sample function $x'(t)$ is related to the received signal $x(t)$ as,

$$x'(t) = x(t) - \sum_{j=1}^{N} x_j \phi_j(t) \rightarrow (5)$$

subs (1) & (2) in (5)

$$x'(t) = s_i(t) + w(t) - \sum_{j=1}^{N} (s_{ij} + w_j)\phi_j(t)$$

$$= s_i(t) + w(t) - \sum_{j=1}^{N} s_{ij}\phi_j(t) - \sum_{j=1}^{N} w_j \phi_j(t)$$

$$= w(t) - \sum_{j=1}^{N} w_j \phi_j(t).$$

$$\hookrightarrow = w'(t) \rightarrow (6)$$

∴ Sample func^n $x'(t)$ depends on channel noise $n(t)$

→ On the basis of eqns (5) & (6), the received signal is defined as,

$$x(t) = \sum_{j=1}^{N} x_j \phi_j(t) + x'(t)$$

$$\hookrightarrow = \sum_{j=1}^{N} x_j \phi_j(t) + w(t) //.$$

* Optimum receivers using coherent detection:-

1. Maximum Likelihood detection:-

→ Suppose that in each time slot of duration 'T' seconds, one of the M possible signals $s_1(t), s_2(t) \dots s_M(t)$ is transmitted with equal probability, $1/M$.

→ The transmitted signal $s_i(t)$ can be represented by a point in a Euclidean space of dimension $N \le M$, this point is referred as the transmitted signal point or message point.

→ The set of message points corresponding to the set of transmitted signal $\{s_i(t)\}_{i=1}^{M}$ is called message constellation.

→ When the received signal $x(t)$ is applied to the bank of 'N' correlators, the correlator o/ps define the observation vector.

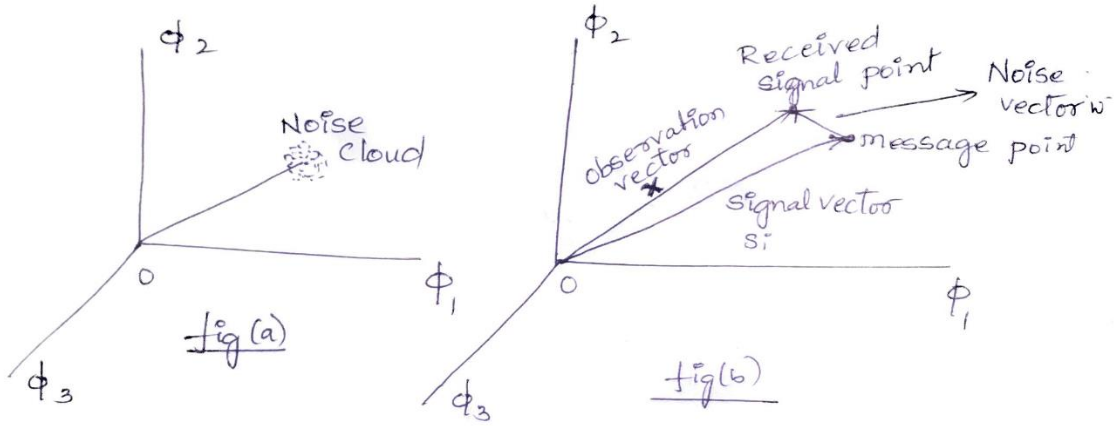
→ W.K.T. $x = s_i + w$ $i = 1, 2 \ldots M$.

The vector 'x' differs from the signal vector s_i by the noise vector w, whose orientation is completely random.

→ Based on observation vector 'x', the received signal $x(t)$ is represented on same Euclidean space & this point is referred as received signal point.

→ Due to the presence of noise, the received signal point moves randomly and lie anywhere inside a Gaussian-distributed cloud as shown in fig.(a)

→ Fig. (b) illustrate the relationship b/w x & s_i



fig(a) fig(b)

→ The signal detection problem is stated as, given the observation vector x, perform a mapping from 'x' to an estimate $\hat{m}$ of the transmitted symbol 'm_i', in a way that would minimize the probability of error in the decision making process.

→ The probability of error is denoted by $P_e(m_i|x)$ & it is given by,

$$P_e(m_i|x) = 1 - P(m_i \text{ sent}|x) \rightarrow (1)$$

→ On the basis of above eq^n, the optimum decision rule is stated as,

Set $\hat{m} = m_i$ if $P(m_i|\text{ sent }|x) \geq P(m_k|\text{sent }|x) \rightarrow (2)$

for all $k \neq i$, & $k = 1, 2 \ldots M$.

Eq^n (2) is referred as the maximum a posteriori probability (MAP) rule & the system used to implement this rule is called a maximum a posteriori decoder.

→ The MAP rule is restate as, set $\hat{m} = m_i$ if

$$\frac{\pi_k f_X(x|m_i)}{f_X(x)} \text{ is maximum for } k = i \rightarrow (3)$$

where, π_k → prior probability of transmitting symbol m_k

$f_X(x|m_i)$ → conditional probability density function of 'X'

$f_X(x)$ → Uncondition probability density function of 'X'

→ In eq^n (3),

* the denominator $f_X(x)$ is independent of the transmitted symbol
* $\pi_k = \pi_i$ when all the source symbols are transmitted with equal probability
* $f_X(x|m_k)$ bears a one to one relationship to the log-likelihood function $L(m_k)$.

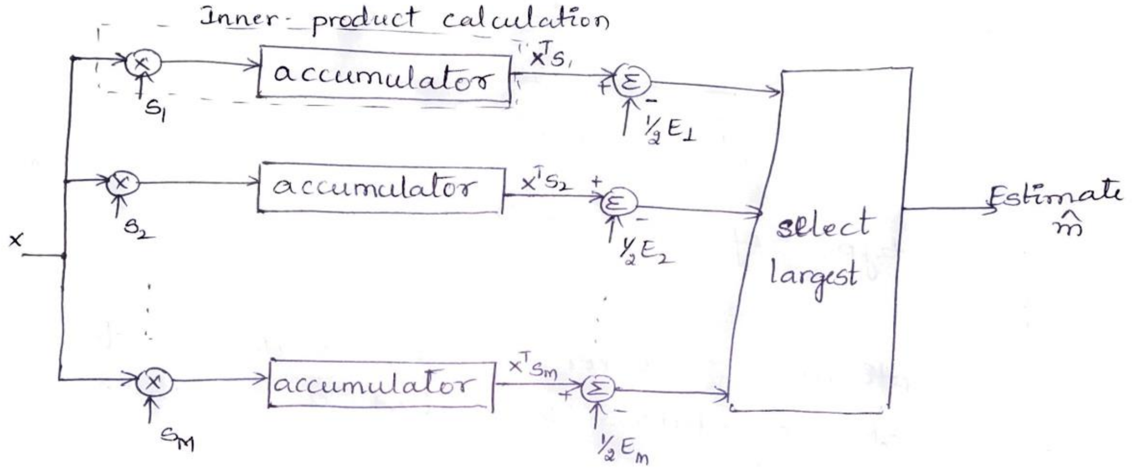
→ Restating MAP rule interms of $L(m_k)$,

set $\hat{m} = m_i$ if $L(m_k)$ is maximum for $k = i$ → (4)

⇒ Eq^n (4) is known as the maximum likelihood rule & the system used to implement this is called maximum likelihood decoder

→ According to this rule, a maximum likelihood decoder computes the log-likelihood functions as metrics for all the 'M' possible the observation vector 'x' by the corresponding N elements of each of the M signal vector $s_1, s_2 \ldots s_M$. Then, the resulting products are successively summed in accumulators to form the corresponding set of inner products $\{x^T s_k \mid k = 1, 2 \ldots M\}$

→ The inner products are corrected for the fact that the transmitted signal energies may be unequal. Finally, the largest one of the resulting set of numbers is selected & the decision is made.



* Matched filter receiver :-

→ Fig(a) can be replace by other equivalent structure, so consider a Linear TI filter with impulse response $h_j(t)$.

→ Let $x(t)$ be received signal which acts as input, the resulting filter o/p is defined by,

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)\, h_j(t-\tau)\, d\tau \longrightarrow (1)$$

Replace limits $0 \le t \le T$ & 'τ' by 't'

$$\therefore\ y_j(T) = \int_0^T x(t)\, h_j(T-t)\, dt \longrightarrow (2)$$

→ Consider a detector based on a bank of correlators. The o/p. of j^{th} correlator is,

$$x_j = \int_0^T x(t)\, \phi_j(t)\, dt \longrightarrow (3)$$

message symbol & then compare them & take decide in favor of the maximum. Thus, the maximum likelihood decoder is simplified version of maximum a posteriori decoder, in that the message symbols are assumed to be equally likely.

* Correlation receiver :-

→ The optimum receiver for an AWGN & for the case when the transmitted signals $s_1(t), s_2(t) \ldots s_M(t)$ are equally likely is called a correlation receiver.

→ It consists of two subsystems as shown in fig,

1. Detector, consists of M correlators supplied with a set of orthonormal basis functions $\phi_1(t), \phi_2(t) \cdots \phi_N(t)$ that are generated locally, this bank of correlators operates on the received signal $x(t)$, $0 \le t \le T$, to produce the observation vector x.

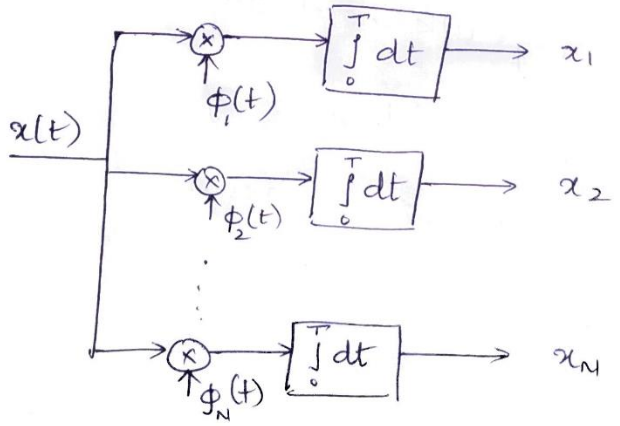


fig.(a):- Detector

2. Maximum likelihood detector operates on observation vector 'x' to produce an estimate $\hat{m}$ of the transmitted symbol m_i, $i = 1, 2 \ldots M$

$$\therefore h_j(T-t) = \phi_j(t) \quad \text{for } 0 \le t \le T \text{ \& } j = 1, 2 \ldots n$$

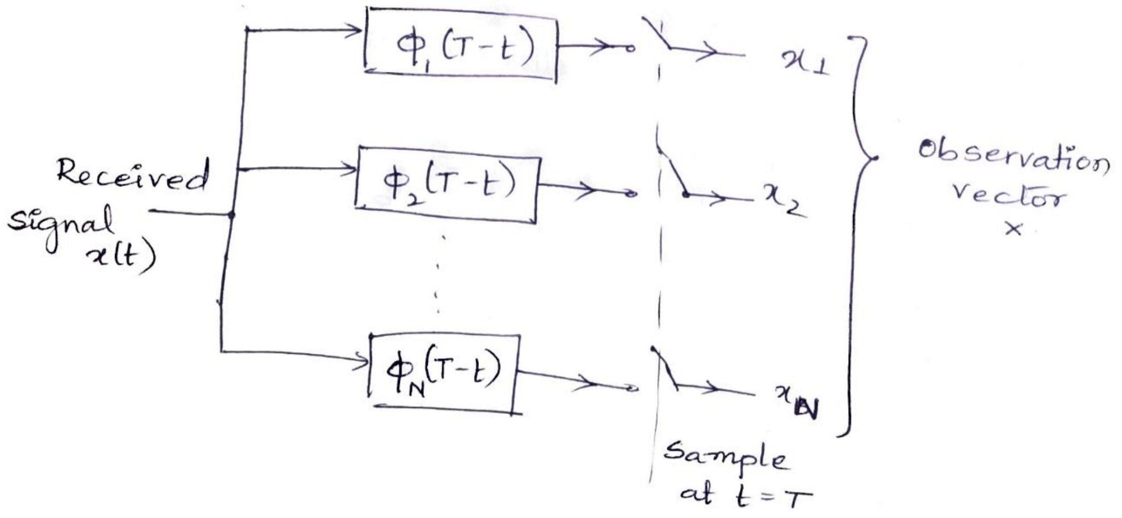
$$(\text{or}) \quad h_j(t) = \phi_j(T-t) \quad \text{for } 0 \le t \le T \text{ \& } j = 1, 2 \ldots n$$

→ Generalising, given a pulse signal $\phi(t)$ occupying the interval $0 \le t \le T$, a LTI filter is said to be matched to the signal $\phi(t)$ if its impulse response $h(t)$ satisfies the condition

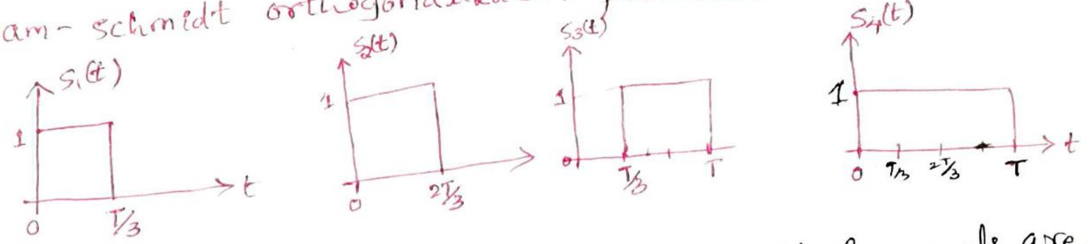
$$\boxed{h(t) = \phi(T-t) \quad 0 \le t \le T}$$

This type of LTI filter is called a matched filter

→ An optimum receiver using matched filters in place of correlators is called a matched filter receiver



(1) consider the signals $s_1(t)$, $s_2(t)$, $s_3(t)$ & $s_4(t)$ as shown in fig. Find orthonormal basis for this signals using Gram-schmidt orthogonalization procedure. (1)



Soln: from fig. $s_4(t) = s_1(t) + s_3(t)$ ∴ this set of signals are not linearly independent.

According to this procedure the signals are linearly independent.

∴ The energy of signal $s_1(t)$ is, $E_1 = \int_0^T s_1^2(t)\,dt$

$$\therefore E_1 = \int_0^{T/3} (1)^2 dt = t\Big|_0^{T/3} = \frac{T}{3}.$$

∴ The first basis function is given by, $\boxed{\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}}$ — (a)

using 'E_1' value in 'ϕ_1'

$$\therefore \phi_1(t) = \frac{1}{\sqrt{T/3}} = \sqrt{\frac{3}{T}} \quad \text{— (1)}$$

$$\boxed{\therefore \phi_1(t) = \begin{cases} \sqrt{3/T} & 0 \le t \le T/3 \\ 0, & \text{elsewhere} \end{cases}}$$

∴ The second basis function is, $\boxed{\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}}$ — (A)

The energy of signal $s_2(t)$ is, $E_2 = \int_0^{2T/3} s_2^2(t)\,dt = \int_0^{2T/3} 1\,dt$

$$E_2 = t\Big|_0^{2T/3} = \frac{2T}{3} \quad \text{— (2)}$$

To find coefficient s_{21},

$$s_{21} = \int_0^T s_2(t)\phi_1(t)\,dt = \int_0^{T/3} s_2(t)\phi_1(t)\,dt + \int_{T/3}^{2T/3} \underbrace{s_2(t)\phi_1(t)}_{\phi_1(t)=0}dt$$

$$= \int_0^{T/3} s_2(t)\phi_1(t)\,dt$$

(2)

$$s_{21} = \int_0^{T/3} 1 \cdot \sqrt{\frac{3}{T}}\, dt = \sqrt{\frac{3}{T}} \int_0^{T/3} 1 \cdot dt$$

$$= \sqrt{\frac{3}{T}} \times \left[\frac{T}{3}\right]$$

$$s_{21} = \sqrt{\frac{T}{3}} \quad \text{— (3)} \quad \text{for all 't'}$$

substituting eqn (2) and eqn (3) in eqn (A)

$\cancel{\phi_2(t) =}$ ~~for $0 \le t \le \frac{2T}{3}$~~ $\sqrt{\frac{3}{T}}$ for $0 \le t \le T/3$

For $0 \le t \le T/3$

$$\phi_2(t) = \frac{s_2(t) - s_{21}\,\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} = \frac{1 - \sqrt{T/3}\sqrt{3/T}}{\sqrt{2T/3 - (\sqrt{T/3})^2}} = 0 \quad \text{For } 0 \le t \le T/3$$

For $T/3 \le t \le 2T/3$,

$$\phi_2(t) = \frac{1}{\sqrt{T/3}} = \sqrt{\frac{3}{T}} \quad \text{For } T/3 \le t \le 2T/3$$

$\left[\because \phi_1(t) = 0,\ s_2(t) = 1,\ s_{21} = \sqrt{T/3} \quad \text{For } \frac{T}{3} \le t \le \frac{2T}{3}\right]$

$$\therefore \quad \phi_2(t) = \begin{cases} \sqrt{\frac{3}{T}} & T/3 \le t \le \frac{2T}{3} \\ 0 & \text{otherwise.} \end{cases} \quad \text{— (4)}$$

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)\,dt}} \quad \text{— (B)}$$

$\left[\because \phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,dt}};\ i = 1, 2, \ldots N\right]$

$$\text{but } g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\,\phi_j(t)$$

For, $i = 3$

$$g_3(t) = s_3(t) - \sum_{j=1}^{3-1} s_{3j}\,\phi_j(t)$$

$$= s_3(t) - \sum_{j=1}^{2} s_{3j}\,\phi_j(t)$$

$$g_3(t) = s_3(t) - s_{31}\,\phi_1(t) - s_{32}\,\phi_2(t) \quad \text{— (5)}$$

$$s_{31} = \int_0^T s_3(t)\,\phi_1(t)\,dt = 0 \quad \text{— (6)}$$

$\left[\because \text{as } s_3(t) = 1 \quad T/3 \le t \le T;\ \phi_1(t) = \sqrt{\frac{3}{T}} \quad 0 \le t \le T/3\right]$

$$\& \quad s_{32} = \int_0^T s_3(t)\,\phi_2(t)\,dt$$

(3)

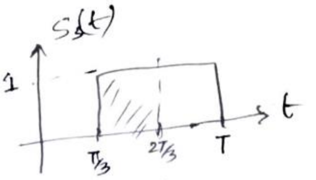
$$s_{32} = \int_0^{T/3} s_3(t)\cancel{\phi_2(t)}^{\,0}\,dt + \int_{T/3}^{2T/3} s_3(t)\,\phi_2(t)\,dt + \int_{2T/3}^{T} s_3(t)\cancel{\phi_2(t)}^{\,0}\,dt$$

$$s_{32} = \int_{T/3}^{2T/3} s_3(t)\,\phi_2(t)\,dt = \int_{T/3}^{2T/3} 1\cdot\sqrt{\tfrac{3}{T}}\,dt = \sqrt{\tfrac{3}{T}}\,\big[t\big]_{T/3}^{2T/3}$$

$$= \sqrt{\tfrac{3}{T}}\,\big[2T/3 - T/3\big] = \sqrt{\tfrac{3}{T}}\times\frac{T}{3}$$

$$s_{32} = \sqrt{T/3} \quad — (7)$$

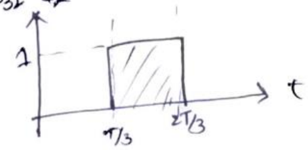
using (6) & (7) in eqn (5).



$$g_3(t) = s_3(t) - \cancel{s_{31}\phi_1(t)}^{\,0} - s_{32}\,\phi_2(t)$$

$$g_3(t) = s_3(t) - \sqrt{T/3}\,\phi_2(t)$$

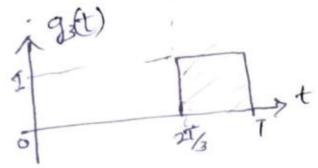
$s_{32}\cdot\phi_2(t) = \sqrt{T/3}\sqrt{\tfrac{3}{T}} = 1; \ \tfrac{T}{3} \le t \le \tfrac{2T}{3}$



$$g_3(t) = \begin{cases} 1, & 2T/3 \le t \le T \\ 0, & \text{otherwise} \end{cases} \quad — (8)$$

eqn (B) can be re-written as,

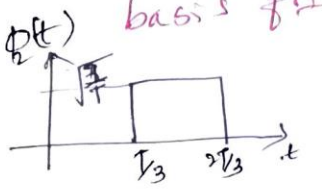
$$\therefore \phi_3(t) = \frac{1}{\sqrt{\int_0^T g_3(t)^2\,dt}} = \frac{1}{\sqrt{\int_{2T/3}^{T} 1^2\,dt}}$$

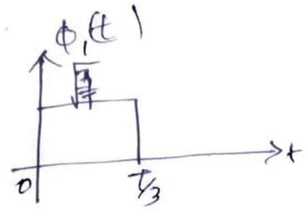


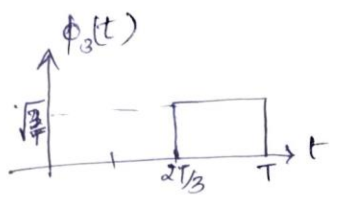
$$\phi_3(t) = \frac{1}{\sqrt{T/3}} = \sqrt{3/T}$$

$$\Rightarrow \boxed{\phi_3(t) = \begin{cases} \sqrt{3/T}, & 2T/3 \le t \le T \\ 0, & \text{otherwise} \end{cases}}$$

Thus $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ forms a set of basis fns.







(4)

further, we can observe that,

$s_1(t) = \sqrt{T/3}\,\phi_1(t)$ $\qquad s_1(t) = s_1 = [\sqrt{T/3}, 0, 0]$

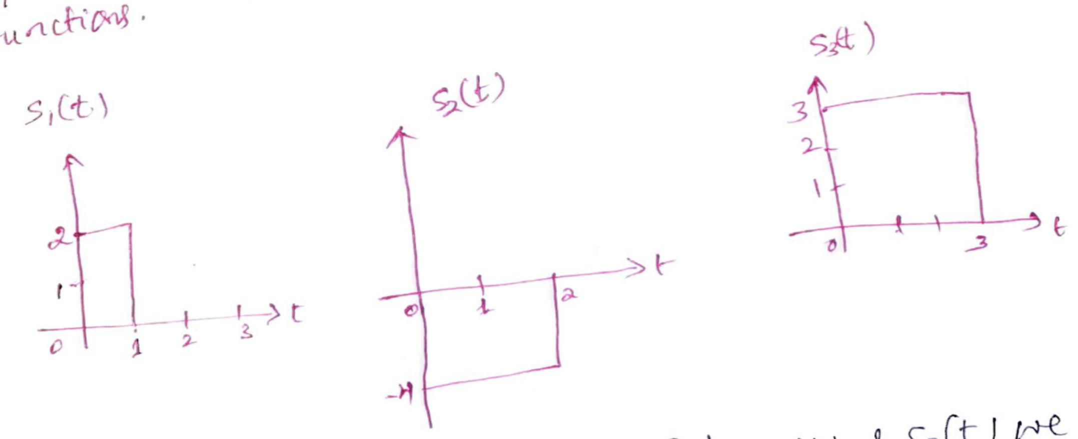
$s_2(t) = \sqrt{T/3}\,[\phi_1(t) + \phi_2(t)]$ $\qquad s_2(t) = s_2 = [\sqrt{T/3}, \sqrt{T/3}, 0]$

$s_3(t) = \sqrt{T/3}\,[\phi_2(t) + \phi_3(t)]$ $\qquad s_3(t) = [0, \sqrt{T/3}, \sqrt{T/3}]$

$s_4(t) = \sqrt{T/3}\,[\phi_1(t) + \phi_2(t) + \phi_3(t)]$ $\qquad s_4(t) = [\sqrt{T/3}, \sqrt{T/3}, \sqrt{T/3}]$

So, we can now conclude that only '3' basis fns are sufficient for representing '4' signals given in this particular example.

(2) Using the gram-schmidt orthogonalization procedure, find a set of orthonormal basis functions to represent the signals $s_1(t)$, $s_2(t)$ and $s_3(t)$ shown in figure below. Also express each of the signals interms of the set of basis functions.



Soln: From the observation of $s_1(t)$, $s_2(t)$ & $s_3(t)$ we can say that the given signals are independent.

$\therefore N = M = 3$. we need to find '3' orthogonal basis functions.

(5).

(i) To find $\phi_1(t)$:

The energy of signal $s_1(t)$ is, $E_1 = \int_0^1 (2)^2 dt$

$$\boxed{\therefore E_1 = 4J} \quad \text{—(1)}$$

w.k.t. $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{2}{\sqrt{4}} = 1$

$$\Rightarrow \boxed{\phi_1(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}} \quad \text{—(2)}$$

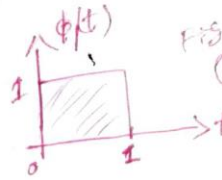


Fig (1)

(ii) To find $\phi_2(t)$:

The energy of signal $s_2(t)$, $E_2 = \int_0^2 (4)^2 dt$

$E_2 = \left[16t\right]_0^2 = \underline{32J}$ —(3)

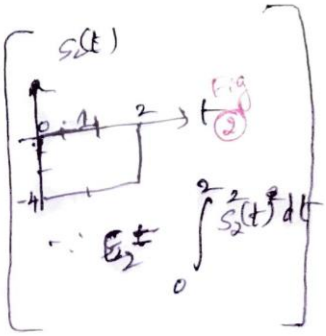


Fig (2)

w.k.t. $\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$ —(4)

To find coefficient s_{21},

$$s_{21} = \int_0^2 s_2(t)\,\phi_1(t)\,dt$$

$$= \int_0^1 s_2(t)\,\phi_1(t)\,dt + \int_1^2 \cancel{s_2(t)\,\phi_1(t)}^{\,0}\,dt$$

$$= \int_0^1 (-4 \times 1)\,dt$$

$$\boxed{s_{21} = -4} \quad \text{—(5)}$$

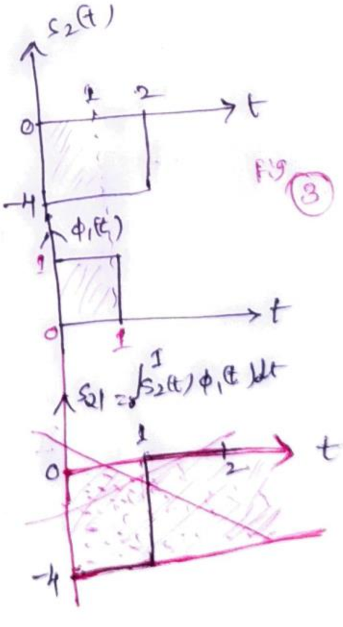
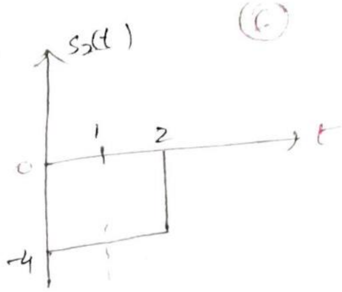


Fig (3)

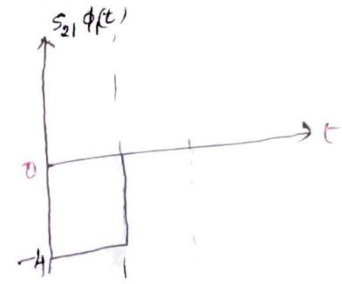
Now, $\phi_2(t)$:

for interval $0 \le t \le 1$



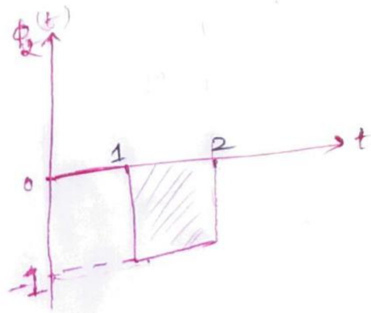
$$\phi_2(t) = \frac{s_2(t) - s_{21}(t)\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

$$= \frac{-4 - (-4)}{\sqrt{E_2 - s_{21}^2}}$$



Fig(4)

$$\boxed{\phi_2(t) = 0 \qquad 0 \le t \le 1}$$ (6)



For interval $1 \le t \le 2$

$$\phi_2(t) = \frac{s_2(t) - s_{21}(t)\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

$$\phi_2(t) = \frac{(-4) - (-4)(0)}{\sqrt{32 - (-4)^2}} = \frac{-4}{\sqrt{16}} = -1$$

$$\therefore \quad \boxed{\phi_2(t) = \begin{cases} -1, & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}}$$ (7)

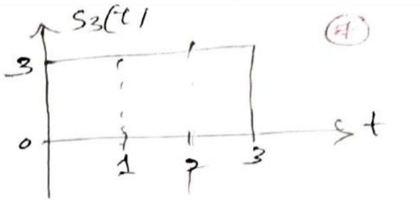
Now, to find $\phi_3(t)$:

let us define an intermediate function $g_3(t)$

$$\left[\because g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j\right]$$

$$g_3(t) = s_3(t) - \sum_{j=1}^{2} s_{ij}\phi_j$$

$$g_3(t) = s_3(t) - [s_{31}\phi_1 + s_{32}\phi_2]$$ (8)

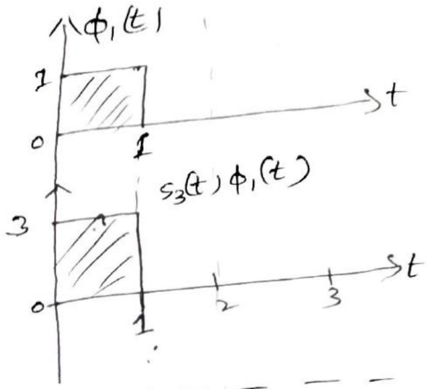


(4)

$$s_{31} = \int_0^T s_3(t)\,\phi_1(t)\,dt$$

$$= \int_0^1 3\,dt$$

$$\boxed{s_{31} = 3} \quad (9)$$

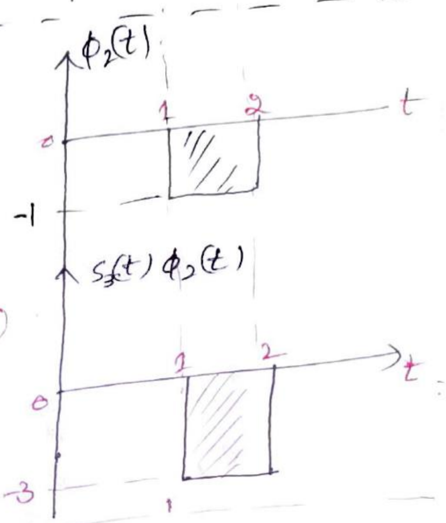


Fig(5)

$$s_{32} = \int_0^T s_3(t)\,\phi_2(t)\,dt$$

$$s_{32} = \int_1^2 (3)(-1)\,dt$$

$$\boxed{s_{32} = -3.} \quad (10)$$



Fig(6)

using 's_{31}' & 's_{32}' values in eqn (8)

$$g_3(t) = s_3(t) - [3\phi_1(t) - 3\phi_2(t)]$$

$$g_3(t) = s_3(t) - 3\phi_1(t) + 3\phi_2(t)$$

Since, $\phi_1(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$

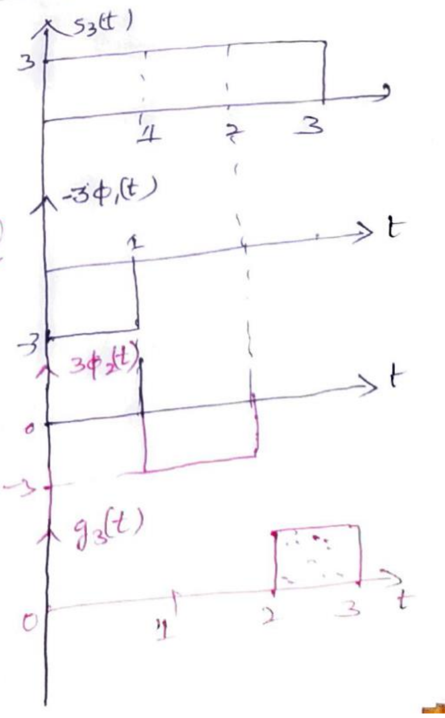
& $\phi_2(t) = \begin{cases} 1, & 1 \le t \le 2 \\ 0, & \text{elsewhere} \end{cases}$

$$\boxed{g_3(t) = 0, \text{ For } 0 \le t \le 2} \quad (11)$$

& For $2 \le t \le 3$

$$g_3(t) = 3 - 0 - 0.$$

$$\boxed{g_3(t) = 3} \quad (12) \quad \text{for } 2 \le t \le 3$$



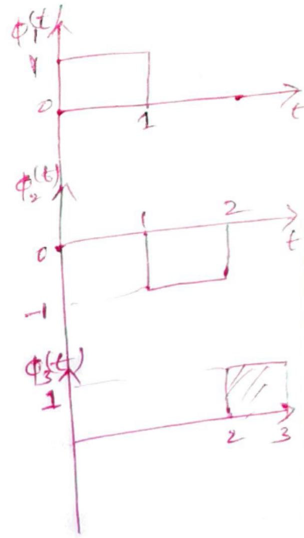
Fig(7)

$\phi_3(t)$ is given by,

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_2^3 g_3^2(t)\,dt}} = \frac{3}{\sqrt{9}} = 1$$

$$\phi_3(t) = \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Now, ~~to~~ ~~find~~ Represent $s_1(t)$, $s_2(t)$ & $s_3(t)$ interms of $\phi_1(t)$, $\phi_2(t)$ & $\phi_3(t)$.



$$s_1(t) = 2\phi_1(t)$$

$$s_2(t) = -4\phi_1(t) + 2\phi_2(t)$$

$$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$$